

In search of a poset structure to the regular exceptional graphs

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ABSTRACT

A (κ, τ) -regular set is a subset of the vertices of a graph, inducing a κ -regular subgraph such that every vertex not in the subset has τ neighbors in it [2].

An exceptional graph is a connected graph with least eigenvalue greater than or equal to -2 which is not a generalized line graph, and in [3] it is shown that the set of regular exceptional graphs is partitioned in three layers. The idea of a recursive construction of regular exceptional graphs is proposed in [1]. With a new technique we prove that all regular exceptional graphs from the 1st and 2nd layer could be produced by this technique.

The new recursive technique is based on the construction of families of regular graphs, where each regular graph is obtained by a (κ, τ) -extension defined by a κ -regular graph H such that $V(H)$ is a (κ, τ) -regular set of the extended regular graph. The process of extending a graph is reduced to the construction of the incidence matrix of a combinatorial 1-design, and these extensions induce a partial order.

Considering several rules to reduce the production of isomorphic graphs, each exceptional regular graph is constructed by a $(0, 2)$ -extension. Based on this construction, an algorithm to produce the regular exceptional graphs of the 1st and 2nd layer is introduced and the corresponding poset is presented, using its Hasse diagram.

CONSTRUCTION OF REGULAR GRAPHS BY (κ, τ) -EXTENSIONS

The procedure that generates the graph $H \oplus G$ from G is called (κ, τ) -extension of

the graph G and the adjacency matrix is $A_{H \oplus G} = \begin{bmatrix} A_H & B \\ B^T & A_G \end{bmatrix}$, where B is the

incidence matrix of a $1-(|V(H)|, \tau, p-\kappa)$ design, that is, each column of B is the characteristic vector of a block. This induces a partial order defined by

If $G, G' \in \mathcal{F}$ then $G \leq G' \Leftrightarrow G'$ can be obtained from G by a sequence of zero or more (κ, τ) -extensions

Applying this procedure recursively we obtain a family of regular graphs

$\mathcal{F} = \{G_m, m \geq 0\}$, with $G_m = H \oplus (H \oplus \dots (H \oplus G))$ a $((p-\tau)+m\tau)$ -regular graph of order $m|V(H)| + |V(G)|$ obtained by a sequence of m (κ, τ) -extensions.

$$A_{H \oplus (H \oplus G)} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

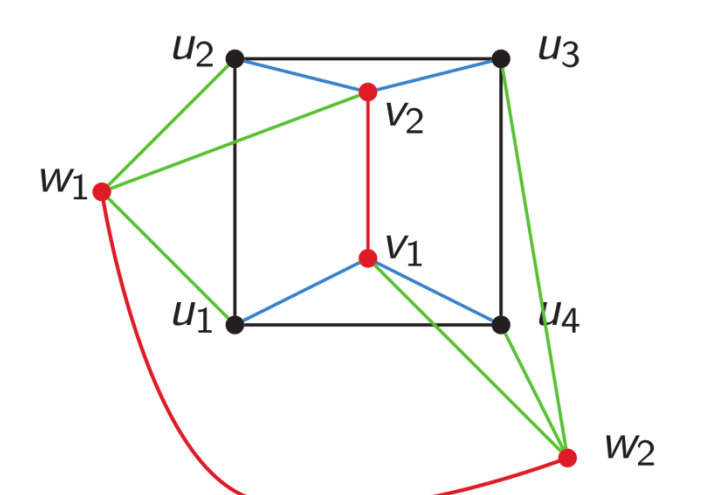


Fig 1. Chain $C_4 \leq K_2 \oplus C_4 \leq K_2 \oplus (K_2 \oplus C_4)$ obtained from C_4 by two $(1, 1)$ -extensions.

CONSTRUCTION OF REGULAR EXCEPTIONAL GRAPHS BY $(0, 2)$ -EXTENSIONS

An exceptional graph G is a connected graph, with $\lambda_{\min}(G) \geq -2$, which is not a generalized line graph. The process of extending a graph by a $(0, \tau)$ -extensions is invariant with respect to the permutation of rows of incidence matrix.

1st layer using $1-(4, 2, n/2)$ designs

Each nonnegative solution of the linear system $\begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} n/2 \\ n/2 \\ n/2 \\ n/2 \end{bmatrix}$ gives the number of columns of the incidence

matrices equal to the characteristic vector s_i . Considering that incidence matrices that differ only by a row permutation generate isomorphic graph we can reduce the set of all solutions to a standard solutions $(x_1, x_2, x_3, x_3, x_2, x_1)$ such that

$$x_1 + x_2 + x_3 = \frac{n}{2}, \quad 0 \leq x_1 \leq x_2 \leq x_3 \leq \frac{n}{2}.$$

Algorithm: starting G_{m-1} ($m \geq 1$), calculate all the standard solutions and generate all the possible B incidence matrices to construct the adjacency matrix of the extended graphs G_m verifying if the $\lambda_{\min}(G_m) \geq -2$.

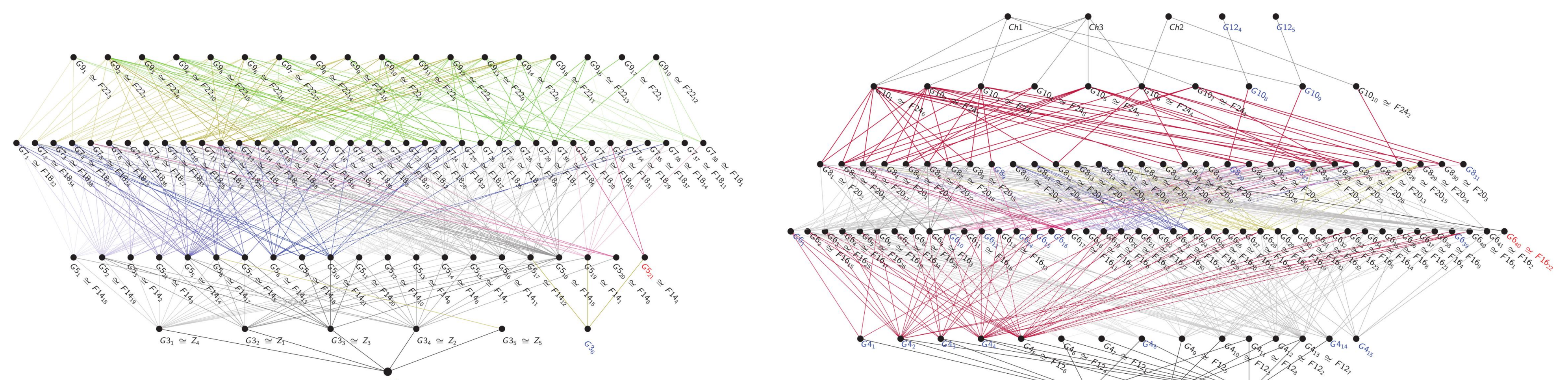


Fig 2. The Hasse diagram of the regular exceptional graphs from the 1st layer is partitioned in two subsets: graphs with odd and even regularities.

2nd layer using $1-(3, 2, 2n/3)$ designs

The solution of the linear system

$$\begin{bmatrix} s_1 & s_2 & s_3 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2n/3 \\ 2n/3 \\ 2n/3 \end{bmatrix} \text{ is } (n/3, n/3, n/3),$$

so $n/3$ columns of the incidence matrix are equal to s_i .

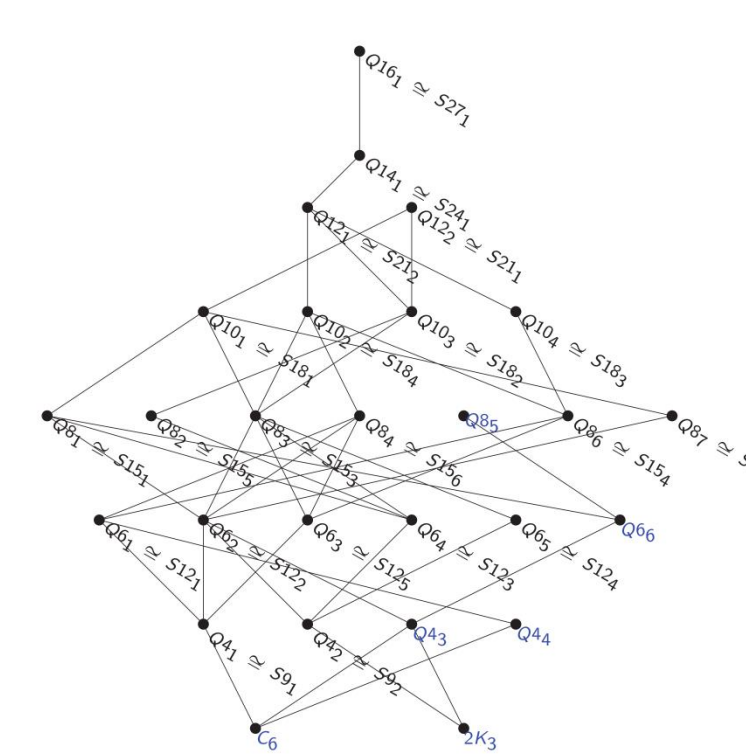


Fig 3. The Hasse diagram of the regular exceptional graphs from the 2nd layer.

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