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Evaluating market pricing competition with the Bertrand Network

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Abstract

Recently in the literature, there have been many attempts to expand classic models of market competition analysis. Considering firms are competing globally against many different sellers over different markets, recent works proposed a model where it is possible to represent competition among companies where they compete against each other directly and indirectly, using a hypergraph to represent the competition structure.

This document presents an attempt to demonstrate how the young and maturing networked price competition model, which allows finding the best price for the companies from the competition structure and market sizes, can be used in any case of study.

This work continues the recent demand to adapt the famous Bertrand competition model, where sellers ask for prices. Since there are no recent works which use the recent model, it has been presented how to use it in such a way that is possible to guess the competition structure and the distribution of the buyers by only by observing how companies are pricing.

To better understand the applications of the existing method, the first real case of study which has used the Bertrand Network model is presented: a competition among 6 flight companies, where prices were collected by using the Google Flight tracking service, concluding that the proofs and claims developed in this work are useful to enhance market analysis.

Keywords: Bertrand network, game theory, Nash equilibrium, market, analysis

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Chapter 1 Introduction

This dissertation emerged after the review of some existing models that apply the Game Theory in the economic field. One new mathematical model that represents price competition with many sellers over many different markets has been chosen to be used for the first time.

This initial chapter aims to present the work developed in its general lines: the validation of a new mathematical model to analyze markets. It presents the project context, motivation, and proposed objectives, as well as a summary of the conclusion and open problems, ending by describing the structure of the rest of the document.

1.1 Project context

Game theory has been applied in the economic scenario since its beginning, becoming one of the main mathematic tools used to analyze the market [1]. Game Theory has the special property of showing the best set of options or probabilities of each player, used mainly to predict social and economic behaviors [2].

Many authors have been trying to build a mathematical model that allows representing competitions where one seller compete against multiple sellers at the same time (directly and indirectly). While made sense to represent only direct competitions in the past, nowadays firms are reacting in real time to price changes with online tools [3], such strategy has been named as Fast-Changing Web Prices [4]. A more robust model is needed to represent a competition where firms are directly and indirectly competing against each other, being recently developed and improved both for games where sellers choose prices [5, 3, 6, 7, 8] and for games where sellers supply an amount of a good [9, 10], leaving the price to be decided by an auctioneer [11].

Although there are many works for the Networked version of the Cournot competition (where sellers supply quantities of a certain good rather than specifically ask for prices, like in the Bertrand competition), the Networked version of the Bertrand model is still young and under development. A thorough search of the relevant literature has not shown any article that has used the Bertrand Network model in a real scenario.

1.2 Motivation

Since many companies are competing globally by having many physical stores around the world or by offering international shipping via online tools, it makes sense to say that companies are competing against many others at the same time. As stated by [6], in the Cournot's famous paper of 1838, zinc producers are not only competing against each other but also indirectly against manufacturers of copper since both have a common major customer: the brass producers. Either zinc and copper producers are targeting the money of the brass manufactures since their materials are needed to produce brass.

Nowadays this happens on a larger scale. As noticed by [4] and quoted by [3], the price of a microwave changed 9 times during the day, a response from Best Buy and Sears to Amazon prices. Motivated by the fact that there is a new mathematical model to represent networked pricing competitions and it lacks practical usages, this work aims to offer an innovative way to approach a new market, study its potential and presents all the steps done for a real case of study.

1.3 Goals

In the existing Bertrand Network model, Nash Equilibrium (N.E.) is found from market sizes, where the Nash Equilibrium represents the best set of pricing range for each seller, as well as the cumulative distribution function (CDF) for each seller, representing its change of asking the price lower than x.

In previous work from the author [8] an unreal competition is used in a reversed manner: prices were said to be collected and it was considered that sellers are being rational in their pricing choices (following their N.E.). Therefore, by having assumed the rationality of the players and by knowing how the equilibrium looks like for some competition structures (from the results in [7]), it was possible to demonstrate how to find the size of the markets from the hypothetic prices and CDFs.

The pricing competition that has been chosen for this work is a competition among 6 air companies and the goal is to apply the steps shown in [8] and figure out the potential of the market, introducing an innovative way to analyze an existing market.

1.4 Contributions of this work

In the present document, the assumptions made in the previous work [8] were extended: since the case of study for air companies has presented a competition network that contains few constraints (companies that asks higher prices must have a bigger captive market than the competitors that are asking lower prices). Also, motivated by some strange results found during the development of this work (like the size of one captive market being negative, which is not possible), assumptions were made and instead of saying that one company is being irrational, it was concluded that hidden competitors (not shown in the research made by the author) must be present, justifying the strange values found.

Moreover, claims have been proved and are presented in the third section of this work, which may enhance the results of the application of the proposed reversed approach.

Theorem 3.1. *Every inverse CDF is upper bounded.*

This theorem is important to show that every inverse CDF is bounded so conditions like $\alpha_i \geq \alpha_{i+1}$ and $\alpha_i > 0$ is true for every seller. From this result, the following corollaries was also proved to be correct.

Corollary 3.1. *Any dataset is called reliable if, for every inverse CDF function calculated from the dataset using (2), its upper bound condition from Theorem 1 is true and every upper bound is lesser or equal than 1. Furthermore, a non-reliable dataset suggests hidden sellers exist in the network or the price data set is too small to be trusted.*

If some conditions from the theorem don't hold, then the dataset cannot be used to calculate the probability of a seller to choose a price of x or higher.

Corollary 3.2. *If an upper bound condition from Theorem 3.1 is greater than 1, there may exist hidden sellers in the network that justify the prices companies are using.*

Corollary 3.3. *If an upper bound condition (9) from Theorem 4.1 is false, there may exist hidden sellers in the network that justify the prices companies are using*

This means that sellers may be hidden in the network. If a condition from the Theorem 3.1 is not satisfied, it is possible that a seller is hidden, which could have made the unsatisfied condition to be true.

Observation 3.1. *Fix any network and any collected prices. It is possible to verify if there exists any size of markets that satisfy the observed prices by testing a non-linear system. Moreover, if the equation system does not have a solution, the network must be changed.*

Observation 3.2. *It is possible to know how much buyers might be willing to pay based on the observed prices*.

Both observations propose a non-linear equation system in which can be used in a software able to deal with this type of equation system.

Open problems extend to show how to calculate the size of the hidden markets, instead of just assuming they must exist.

1.5 Document structure

Section 2 presents the current state of the literature regarding the use of mathematical models which uses Game Theory as the main tool to analyze market competitions, also having a subsection to explain the model with a warm-up example to better understand how it works. Section 3 shows the application of the reversed model in the same hypothetic model to understand its potential, having one subsection with the profs of the claims briefly shown in the previous subsection. Section 4 presents the research methodology used in this work, serving as a bridge from the related work section and the section with the case of study. Section 5 is reserved for the case of study that considered a real competition in the air transport industry, in which the profs and claims from section 3 had to be carefully used. Last, but not least, Section 6 presents the conclusion and open problems.

Chapter 2 Related work

This chapter presents related works that have contributed to this work, either by providing a starting point or by introducing concepts to the author that helped to understand what the scientific community has been recently working on. It is also shown a hypothetic application of the model to better understand its structure.

2.1 Game theory

Game theory presents a mathematical model for decision-making that can be applied in the real world, firstly introduced in 1994 [1]. A game is defined as something composed by a set of players, they can be persons, objects, companies, everything capable to make decisions, simultaneously or sequential to other players. Each movement (or strategy) may have different outcomes, either good or bad. If each player wants to get the best out of the game, each of them must consider the set of options of all the other competitors. It is said that, if a player is always choosing the best option, it is said that the player is being rational [12].

There are many types of games, especially focused on the economic field. Reference [1] presents a case of study from an economic region of Taiwan. Considering local governments with different interests it is proposed a mathematical model for a cooperative game (every player tries to optimize the outcome of both at the same time) and a non-cooperative game, where each of them tries to selfishly optimize their own outcome. The authors concluded that cooperation between them is very beneficial, ending with a suggestion of how they could cooperate. The usage of Game Theory to show that cooperation is possible also appeared in other studies [13, 14]. Another study has also shown that merge of companies can be profitable [10].

Games can be applied either for the external process (competition among firms, for example) or for the internal process of a company. A study has been presented by [14] where it has been constructed a **payoff matrix**, representing the gains and losses of each department for each third-party company that could be chosen. In the end, the Nash Equilibrium has been found, concluding which outsource company should be hired. In the study, there were three departments, each of them with a different priority: low cost, quality and low cost of transport.

Game Theory can also be used to assist banks in a decision-making game: accept or decline loan requests [15]. The most interesting part of this study is that they point out the possibility to use Game Theory to enhance data-mining (analysis of a large dataset with the objective of extracting implicit information).

2.1.1 Nash Equilibrium

Consider two persons on a phone call that ends abruptly. There is no other way to communicate, what should each of them do? If both decide to wait, no one will receive a call, if both decide to call back, the line will be busy and again, they won't be able to communicate. The game is represented in the following **payoff matrix**:

Table 1: Payoff matrix. Rows and columns represent the set of options available for player A and B, respectively.

	Player B	
Player A	Wait	Call back
Wait	0	1
Call back	1	0

As it is possible to see in Table 1, both players have the outcome of zero when they take the same decision. Considering that they are playing at the same time, player A's reasoning will look something like: "*I should wait if he decides to call me back, but what if he is thinking that I'll call him back? Then I should call him back, but what if he is thinking the same? If that is the case, he will call me back, therefore I must wait, but what* *if he is thinking the same? Then…*". How to solve this endless thinking?[1][2] Nash Equilibrium[3] has the goal to square endless thoughts, being the starting point of almost every game [2].

Reference [2] states the **Nash Equilibrium** as being a strategy setting where each player is choosing the best response to what he believes the other player will do in the game. There may be many N.E., in those cases more information is needed to determine which of them is more likely to arise.

Pure Nash Equilibrium. This type of N.E. has the property of, given the set of options of each player, there exists at least one option that yields the best outcome to the player, no matter what the others will do[4].

Consider the competition network in Figure 1. Firms F_1 and F_2 are charging the buyers C_A and C_B to cross their bridge to The Irish Pub. Both firms announce their price simultaneously and the edge from the costumer to the firm represents from which firm they can buy. C_B costumers always buy from the firm that is asking the lower price.

There are many reasons why the costumers C_A won't use the bridge from the firm F_2 (they are loyal to the firm F_1), for instance, they could have had problems with the other firm and so they refuse to pay for their service or maybe the other bridge is just too far or too difficult to reach.

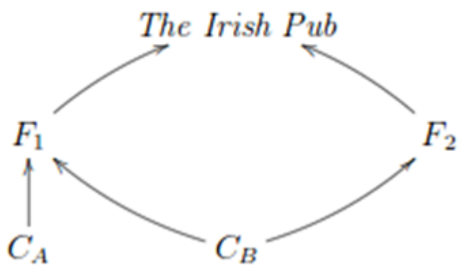

Figure 1: Competition between two toll companies for two sets of consumers. (Reproduced from [5].)

[1] It is possible to argue that there exists an implicit agreement that the person who will be calling back will be the same one who started the call. In fact, this is an example taken from the book: **The Art of Strategy** [2] and it is also addressed in similar situations, where even when a communication is impossible, it is obvious that the decision of the other player will be X or Y. (It is worth reading one of the examples, p. 110.)

[2] A similar example (with endless thinking) can be found at [2], p. 102, being solved with the **Mixed Nash Equilibrium** at the p. 159.

[3] Theorem proved by John Nash, receiving later the Nobel Prize in economics in 1994, the first Nobel Prize for the Game Theory [2]. His work has inspired the movie **A Beautiful Mind**.

[4] In many games it is considered that players are rational. This means that they will not choose one available option if clearly another one would lead to a better outcome. See [2], p. 70 about the usage of dominant strategies and how this characterizes the Nash equilibrium.

It is proven for the given example that there is no **Pure N.E.** Imagine that F_2 is told the price that F_1[5] is going to choose, then, to raise his chance of winning all the buyers C_B he will ask a price arbitrary lower than F_1. The first firm may be able to predict this and therefore will try to undercut the second firm. This would happen until both reaches the marginal cost (in this example, the price zero), therefore both would have no profit. Since F_1 has a captive market (loyal buyers), instead of asking the price of 0 he would choose the maximum price that C_A are willing to pay. F_2, by predicting the movement of F_1, would ask a price arbitrary lower to extract some surplus from the C_b and this reasoning will never halt. There is no price that they would stop changing to obtain a higher profit, a probabilistic feature is needed so they can choose price randomly, tricking the opponent and forcing him to also choose his price in the same way (a **Mixed N.E.**).

If C_A were removed from the competition network, F_1 would have no captive market. Therefore, they would undercut each other until both reaches a marginal cost. A **Pure N.E.** exists since F_1 would not be able to charge a monopolized price from his loyal buyers ($C_A = 0$). If any of the firms decide to charge a higher value, they would for sure lose C_B to the competitor, therefore none of the firms are willing to raise the price by any reason, even if the competitor decides to do so. Charging a lower price does not worth it since they would have a deficit. It doesn't matter what the other firm will do, choosing the price of 0 will always the best option[6].

Mixed Nash Equilibrium. As briefly shown before, Mixed N.E. solve games where Pure N.E. does not exist by adding probability features. As shown in [16] after analyzing 459 penalties they concluded that it is impossible for the goalkeeper to have a strategy that guarantees he will always catch the ball. In that game, the ball can travel 11 meters from the kick mark to the goal at a maximum speed of 125 mph, reaching the goal about two-tenths of a second after having been kicked. Therefore, it is possible to safely say that they must decide their actions before the kick.

[5] The idea that one company may know what the competitor will do comes from the fact that, considering each player are rational and they are trying to maximize their utility level, both tries to predict what the other will do, therefore they try to be one step forward. The idea of being one step forward is represented in [2], chapter 5 with the game **The Princess Pride**, where the hero (Westley) and the villain (Vizzini) plays a game: the hero poison one of the two cups, takes one to himself and gives the other to Vizzini that now has a choice of swapping or not the cups. The problem comes with the endless thinking of both players, the hero might have tried to trick the opponent by taking the poisoned cup to himself, expecting to be one step forward by imaging the adversary will swap the cups. But what if Vizzini is 1 step ahead the hero? The hero may try to be 1 more step ahead. This can go infinitely and the best option for both players is to decide to give or swap the cups randomly with equal probability. Unfortunately to the villain, the hero had poisoned both cups because he is immune to the poison that was being used: they were playing different games without knowing.

[6] It is assumed that both firms cannot communicate and agree to charge the same price, forming a cartel. In many countries, this is forbitten [2].

The study concludes that the goalkeeper must jump to the kicker's left more frequently than kickers choose to kick. In fact, this is what happens: goalkeepers choose to jump to the kicker's natural side 56.6 percent of kicks, compared to 44.9 percent instance for kickers[7].

The most important fact about NE is that, for any finite and continuous game, there exists at least one pure or mixed NE [17].

Non-Nash equilibrium. It is worth noting the difference between a Nash and a Non-nash equilibriums. A Non-NE is basically a strategy that can be chosen that will lead the player to regret his decision. When a player is behaving rationally, he cannot choose a strategy A that, knowing the adversary expectations, would lead to a worse outcome than choosing a strategy B

Consider the game Rock, Paper and Scissors. There is no pure NE, it is not possible to choose one of the 3 options without regretting the decision, there is always a better option to be chosen. Imagine player A always choose paper, player B will be able to predict his next moves and start choosing scissors. In this example, player A would be forced to change his original strategy. This doesn't happen in equilibrium.

In this game, players must randomize and choose each strategy with 33.33% of probability [2], where each player would win roughly 33.33% of the time, having the same chance for drawing or losing. The mixed NE shows the best set of probability for each player. If player A decides to choose rock with 1% of probability instead of 33.33%, he will only be benefiting player B because every time B plays scissor (33.33% of the time), he can or have a tie, or have a win, which is great for B but not for A. In other words, not very mixed probability represents an equilibrium.

2.2 Competition Models

There is two famous competition model in the literature, both having been proposed for duopoly instances and being expanded in future works: the Cournot and the Bertrand model. Both models were published in the nineteenth century and are still being used in

[7] In the study the center has been ignored since it represented less than 3% of the dataset.

the economic field. The remainder of this section explains each of the models, followed by some works from other authors related to those models.

2.2.1 Cournot model

In the model from 1838, the competition among firms occurs by them supplying an amount of a homogeneous good (identical), while the price is determined by an auctioneer [11]. This idea may not look trivial, but it has been proposed considering two sellers that possessed one source of water each, supplying to the market water instead of charging a price per volume, being the first model that considered a duopoly competition [9].

There are many papers related to this model, in particular considering electricity supply companies [18, 19, 20, 21, 22, 23], being one of the most common model used to represent competition among few firms, getting better results than other complex methodologies such as the Supply Function Equilibria (SFE)[8].

There are methods that try to optimize the SFE, one of them is known as the Cournot Adjustment Process [24]. Basically, each competitor asks his marginal cost one at a time. The next, by observing other's offers, performs an optimization step, converging rapidly to the SFE equilibrium.

Recently it has been proposed an algorithm that can compute the NE in a competition network [9]. In the classic model, firms compete *à la Cournot* for only one market. However, as briefly stated in the first section of this paper, it is common that sellers are competing against multiple sellers, either directly and indirectly (See Figure 2). Therefore, a model capable of representing this kind of competition is needed and has been explored in the past few years [25, 26, 27]. As shown in [28], the efficiency loss changes when you consider a competition network where not every company disputes for every market (discriminatory access), instead of considering that every company participates in every market. The authors state that the networked competition model is still maturing and there are few works available.

[8] Model where it is considered that every competitor is trying to maximize their gains under a demand uncertainty [24]. However, this model requires an elevated computational complexity ($O(2^n)$ $possibilities$) and it in some cases a solution may not exist. Therefore, many studies considered only small competitions.

Figure 2: Competition between 10 multinationals competing with one another in various product categories. (Reproduced from [29].)

2.2.2 Bertrand Model

Bertrand, in 1883, after having criticized the Cournot model, he developed a duopoly model where firms compete by asking prices, instead of supplying an amount of good to the market. This is, in fact, a contrast to the criticized model and he concluded that both companies would try to undercut each other until they reach their marginal cost, the minimum price they can ask that pays the production expenses [9].

This model has been used in many competition studies based on prices [3]. In [30] it was suggested a mathematical model where firms compete with prices and costumers decides how many of the given good they want from each firm, while every firm has a limit of how many items they can sell. This model was used in the study of Cognitive Radio Networks [31], where there were many buyers willing to buy bandwidths. In the game, sellers must place the distribution towers to have no interference, but also considering how do the buyers are located.

In the following subsection, it is presented the Bertrand Network model, a maturing mathematical model that can be used to represent direct and indirect competitions. In the existing models, big sellers influence other sellers in their price decision even if they are not direct competitors.

2.3 Bertrand Network

This subsection presents a model for price competition in networked markets firstly introduced by [5] and expanded in other works [7, 3, 6, 8]. The first model contains a very basic structure, while the expanded model contains proofs of the existence of NE for any competition network. As stated earlier, John Nash has proven that every game has a Nash Equilibrium but there are some requirements for those games: the game, the number of players and the set of strategies must be finite. Also, for any sharing rule, “if the strategy spaces are compact metric and the payoff correspondence is bounded and upper hemi-continuous, with nonempty, compact, convex values, then such a solution exists” [32].

For this work, it was considered the model in [7] since its structure is more robust, also containing profs and demonstration for some types of competitions. The most important contribution in [7] is that the authors assert the existence of the NE in any setting of their price competition game by proving that their game is an instance of a Discontinuous Game with Endogenous Sharing Rules, proven to always have an NE [32]. In their case, discontinuous utility[9] arises when a tie occurs: two sellers are asking the same price. To exist an NE in any network, the authors simply consider the subspace of strategies where there are no ties. Therefore, it is not needed to explicitly say what happens at the tie points since it won’t interfere in the mixed NE.

Extensions of the model were not used because they are just improvement of the utility formula from [7] and some assumptions were relaxed, they do not change the general idea of the model (like considering a cost formula to the utility of the sellers, which turns out to be only a constant that decreases the price value [3]). Since this work attempts to find an application for this model, it was focused in the second version of the networked model.

Now, the following properties of the model:

[9] Can also be understood in this context as “profit”

- Let $H = (F, M)$ be the hyper-graph (graph, for short) where F are firms and M represents the markets. The market m_k groups any firm f_i that participates in the market k.
- For this model, $|m_k| = 2$ for any k. This means that only two firms can compete for a market share. For every m_k that represents the competition between the firms f_i and f_j, $i \neq j$, the size of the market share (number of buyers) is represented by $\beta_{i,j} > 0$.
- Each firm i may have a captive market[10] of size $\alpha_i > 0$.
- Firms cannot price discriminate, this means they offer the same price for every market he participates.
- Let $N(i)$ be the set of neighbors of i.
- Every buyer $\beta_{i,j}$ buys from the firms that asks the lowest price.
- Buyers are willing to pay a price that ranges from 0 to 1 (inclusive).
- The utility function for the firm i is:

$$u_i(x) = x\left(\alpha_i + \sum_{j\in N(i)} \beta_{i,j} \cdot \overline{F}_j(x)\right) \tag{1}$$

- $\overline{F}_i(x)$ is a non-decrescent function that represents the inverse Cumulative Distribution Function, the chance of firm i to choose the price of **at least $\boldsymbol{x}$** (price greater than or equal to x). It may be understood as a portion of the market that will buy from them at the price x.
- Marginal costs for every firm are set to 0.
- In the game, firms choose an interval of prices. This means that, for an interval $[t_m, t_n]$, the firm does not do better than randomizing a price within his support (strategy). The support of each firm is in $[0,1]$ and is refered as S_i. Firms may have multiple intervals.

[10] Monopolized market. Can be understood as a set of buyers that will buy from the firm no matter the price, as long it is lower or equal than 1. There are several reasons for this to happen: bad experience with other sellers, geographic reasons, etc.

- When a firm is said to have an atom[11] in 1, the firm chooses the price of 1 with positive probability. In other words: $\overline{F}_i(1) > 0$. Only firms with a positive captive market may have an atom at 1.

Example of three sellers in line. To better understand the model, a warm-up example is introduced for a hypothetic competition.

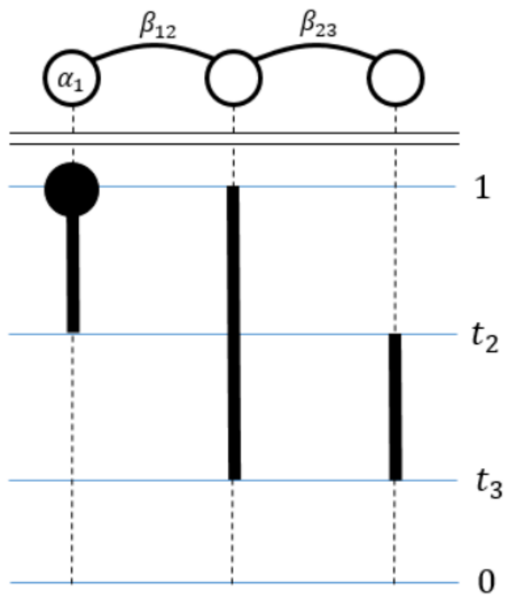

Figure 3: Line competition network. Empty nodes represent firms with no captive market. Firms 1 to 3 are represented from left to right. The support of each firm is represented with the thick lines and the black circle represents an atom in 1. (Reproduced from [7].)

In any equilibrium, each firm must be indifferent in extracting the maximum surplus from his captive market and conquer all the buyers from the market shares. According to the results in [7], firm 1 has the support of $[t_2, t_1]$ because at t_1 he sells only to the captive market, and at the value t_2, he sells both to the whole market share $\beta_{1,2}$ and to his loyal buyers α_1. To be unpredictable, firms must mix, thus the inverse CDF function that says to the firms how to randomize.

On the other hand, firm 2 must be indifferent between extracting the surplus from both market shares. There exists an interval $[t_2, t_1]$ and $[t_3, t_2]$ that allows the second seller to compete for both markets $\beta_{1,2}$ and $\beta_{2,3}$ in a balanced way.

[11] The word atom comes from the measure theory, which means that, for a given measure of a value, the result is positive. This is the case when it is measured the probability of a firm to choose the price of 1. If the measure (result) yields a positive value, then that value has an atom.

Last, but not least, firm 3 must compete in the interval $[t_3, t_2]$ because he will always benefit if firm 2 randomly choses any price in $[t_2, t_1]$, wining the whole market $\beta_{2,3}$. Considering that firms are playing rationally; this reasoning will affect how the second seller will choose any price in $[t_2, t_1]$.

All the statements above can be formalized as follows:

$$u_1(t_1) = u_1(x) \text{ for all } x \in [t_2, t_1]$$

$$u_2(t_1) = u_2(x) \text{ for all } x \in [t_3, t_1]$$

$$u_3(t_2) = u_3(x) \text{ for all } x \in [t_3, t_2]$$

To find the to find the equilibrium of the system, it is needed to calculate every price t_k and find the CDF $\overline{F}_i(x)$ for every i and x. The steps are shown below.

Finding t_1 and t_2. From (1):

$$t_1 * \alpha_1 = u_1(t_1) = u_1(t_2) = t_2\left(\alpha_1 + \beta_{1,2}\overline{F}_2(t_2)\right)$$

The best value for t_1 is 1, maximizing the utility of f_1 at t_1.

From the last Equation:

$$\alpha_1 = t_2 * \left(\alpha_1 + \beta_{1,2}\overline{F}_2(t_2)\right) \Rightarrow t_2 = \frac{\alpha_1}{\alpha_1 + \beta_{1,2}\overline{F}_2(t_2)}$$

Finding $\overline{F}_3(t_j)$ for every j in the system. Since the price of t_3 is the minimal price in the support of f_3, $\overline{F}_3(x) = 1$ for all $x \leq t_3$. On the other hand, with t_2 being the maximum price in his support, $\overline{F}_3(t_2) = 0$ by claiming the **lemma 4.2** in [7, p. 9] which states that "no two sellers who share a market both have an atom at the same positive price". This implies that $\overline{F}_2(t_2) > 0$.

Considering that t_1 does not belong to the support of f_3, $\overline{F}_3(t_1) = 0$**.** More generally, $\overline{F}_3(x) = 0$ for all $x \geq t_2$.

Finding $\overline{F}_2(t_j)$ for every j. As $\alpha_2 = 0$, the firm f_2 cannot have an atom at 1, therefore, $\overline{F}_2(t_1) = 0$.

Applying the same reasoning from $\overline{F}_3(t_3)$, $\overline{F}_2(x) = 1$ for all $x \leq t_3$.

Finally, for $\overline{F}_2(t_1)$:

$$t_2 * \beta_{2,3} * \overline{F}_2(t_2) = u_3(t_2) = u_3(t_3) = t_3 * \beta_{2,3} * \overline{F}_2(t_3)$$

$$t_2 * \beta_{2,3} * \overline{F}_2(t_2) = t_3 * \beta_{2,3}$$

$$\overline{F}_2(t_2) = \frac{t_3}{t_2}$$

Further on it is shown that $t_3 = \frac{t_2 * \beta_{1,2}}{\beta_{1,2} + \beta_{2,3}}$:

$$\overline{F}_2(t_2) = \frac{t_3}{t_2} \Rightarrow \overline{F}_2(t_2) = \frac{\frac{t_2 * \beta 1{,}2}{\beta_{1,2} + \beta_{2,3}}}{t_2} \Rightarrow \overline{F}_2(t_2) = \frac{\beta_{1,2}}{\beta_{1,2} + \beta_{2,3}}$$

Finding $\overline{F}_1(t_j)$ for every j. The minimum price in the support of f_1 is t_2, therefore, $\overline{F}_1(x) = 1$ for all $x \leq t_2$.

For $\overline{F}_1(t_1)$:

$$t_1\left(\beta_{1,2}\overline{F}_1(t_1) + \beta_{2,3}\overline{F}_3(t_1)\right) = u_2(t_1) = u_2(t_2) = t_2\left(\beta_{1,2}\overline{F}_1(t_2) + \beta_{2,3}\overline{F}_3(t_2)\right)$$

$$\beta_{1,2} * \overline{F}_1(t_1) = u_2(t_1) = u_2(t_2) = t_2 * \beta_{1,2}$$

$$\overline{F}_1(t_1) = t_2$$

Finding t_3. The last price to be found:

$$t_1\left(\beta_{1,2}\overline{F}_1(t_1) + \beta_{2,3}\overline{F}_3(t_1)\right) = u_2(t_1) = u_2(t_3) = t_3\left(\beta_{1,2}\overline{F}_1(t_3) + \beta_{2,3}\overline{F}_3(t_3)\right)$$

$$\beta_{1,2} * t_2 = u_2(t_1) = u_2(t_3) = t_3(\beta_{1,2} + \beta_{2,3})$$

$$t_3 = \frac{t_2 * \beta_{1,2}}{\beta_{1,2} + \beta_{2,3}}$$

Now it has been found every t_j and $\overline{F}_i(t_j)$ for every i and j. It is possible to find the generic function $\overline{F}_i(x)$ for every x in the support of the firms.

CDF for $\overline{F}_2(x)$. Considering that the support of the firm f_2 is $[t_3, t_2] \cup [t_2, t_1]$, it is shown below the generic formula that passes through every t_j.

$$t_1\left(\alpha_1+\beta_{1,2}\overline{F}_2(t_1)\right)=u_1(t_1)=u_1(x)=x\left(\alpha_1+\beta_{1,2}\overline{F}_2(x)\right)\forall x\in[t_2,t_1]$$

$$\alpha_1=x\left(\alpha_1+\beta_{1,2}\overline{F}_2(x)\right)$$

$$\overline{F}_2(x)=\frac{\alpha_1-x\alpha_1}{x\beta 1,2}$$

$$\overline{F}_2(x)=\frac{\alpha_1}{\beta 1,2}\left(\frac{1}{x}-1\right)\forall x\in[t_2,t_1]$$

Note that, at the boundary points, the values of the inverse CDF for f_2 are the same as found previously for t_1 and t_2.

For the interval $[t_3,t_2]$:

$$t_2\beta_{2,3}\overline{F}_2(t_2)=u_3(t_2)=u_3(x)=x\beta_{2,3}\overline{F}_2(x)\ \forall x\in[t_3,t_2]$$

$$t_2\overline{F}_2(t_2)=x\overline{F}_2(x)$$

$$t_2*\frac{t_3}{t_2}=x\overline{F}_2(x)$$

$$\overline{F}_2(x)=\frac{t_3}{x}\ \forall x\in[t_3,t_2]$$

CDF for $\overline{F}_1(x)$. For all $x\in[t_2,t_1]$, the following holds:

$$t_1\left(\beta_{1,2}\overline{F}_1(t_1)+\beta_{2,3}\overline{F}_3(t_1)\right)=u_2(t_1)=u_2(x)=x\left(\beta_{1,2}\overline{F}_1(x)+\beta_{2,3}\overline{F}_3(x)\right)$$

$$\beta_{1,2}\overline{F}_1(1)=x\beta_{1,2}\overline{F}_1(x)$$

$$t_2=x\overline{F}_1(x)$$

$$\overline{F}_1(x)=\frac{t_2}{x}\ \forall x\in[t_2,t_1]$$

CDF for $\overline{F}_3(x)$. For the last inverse CDF, the following holds for all $x\in[t_3,t_2]$

$$t_2\left(\beta_{1,2}\overline{F}_1(t_2)+\beta_{2,3}\overline{F}_3(t_2)\right)=u_2(t_2)=u_2(x)=x\left(\beta_{1,2}\overline{F}_1(x)+\beta_{2,3}\overline{F}_3(x)\right)$$

$$t_2\beta_{1,2}=x\left(\beta_{1,2}+\beta_{2,3}\overline{F}_3(x)\right)$$

$$t_2\beta_{1,2} = x\beta_{1,2} + x\beta_{2,3}\overline{F}_3(x)$$

$$\overline{F}_3(x) = \frac{t_2\beta_{1,2} - x\beta_{1,2}}{x\beta_{2,3}}$$

$$\overline{F}_3(x) = \frac{\beta_{1,2}}{x\beta_{2,3}} * (t_2 - x)\forall x \in [t_3, t_2]$$

The results have been summarized in the following table:

Table 2: Results summary for each inverse CDF per each interval.

Interval	$\overline{F}_1(x)$	$\overline{F}_2(x)$	$\overline{F}_3(x)$
$[t_{2,}1]$	$\frac{t_2}{x}$	$\frac{\alpha_1}{\beta 1,2}\left(\frac{1}{x} - 1\right)$	0
$[t_3, t_2]$	1	$\frac{t_3}{x}$	$\frac{\beta_{1,2}}{x\beta_{2,3}} * (t_2 - x)$
$[0, t_3]$	1	1	1

Applying the results in a hypothetic example. Consider a competition with the same settings shown in Figure 3. Every firm sells laptops and there are two types of costumers: Whom seeking high-end laptops and others interested a good laptop for casual usage [8].

Seller f_1 sells gaming laptops and f_2 sells computers that are roughly good for games and very good for casual usage. Last, f_3 sells devices not very good for games but does a good job for other type of usage. Let $\beta_{1,2}$ be the number of buyers interested in high performance computers and $\beta_{2,3}$ the portion of users interested either in computers with good performance for games or casual laptops.

Every buyer is willing to pay up to 1000€. f_1 is known to provide computers with high-quality. Let α_1 be the number of buyers interested only on computers from f_1. Other companies are new in the market, having no loyal buyers.

For the values $\alpha_1 = 20$, $\beta_{1,2} = 150$ and $\beta_{2,3} = 185$:

$$t_1 = 1 \rightarrow 1000€$$

$$t_2 = 0.2294520548 \rightarrow 229.94€$$

$$t_3 = 0.102739726 \rightarrow 102.74€$$

The next figures show the inverse CDF the utility of every firm. Formally, $u_1 = 20$ for all x in $[t_2, t_1]$, $u_2 = 34.41780822$ for all x in $[t_3, t_1]$ and $u_3 = 19.00684932$ for all x in $[t_3, t_2]$.

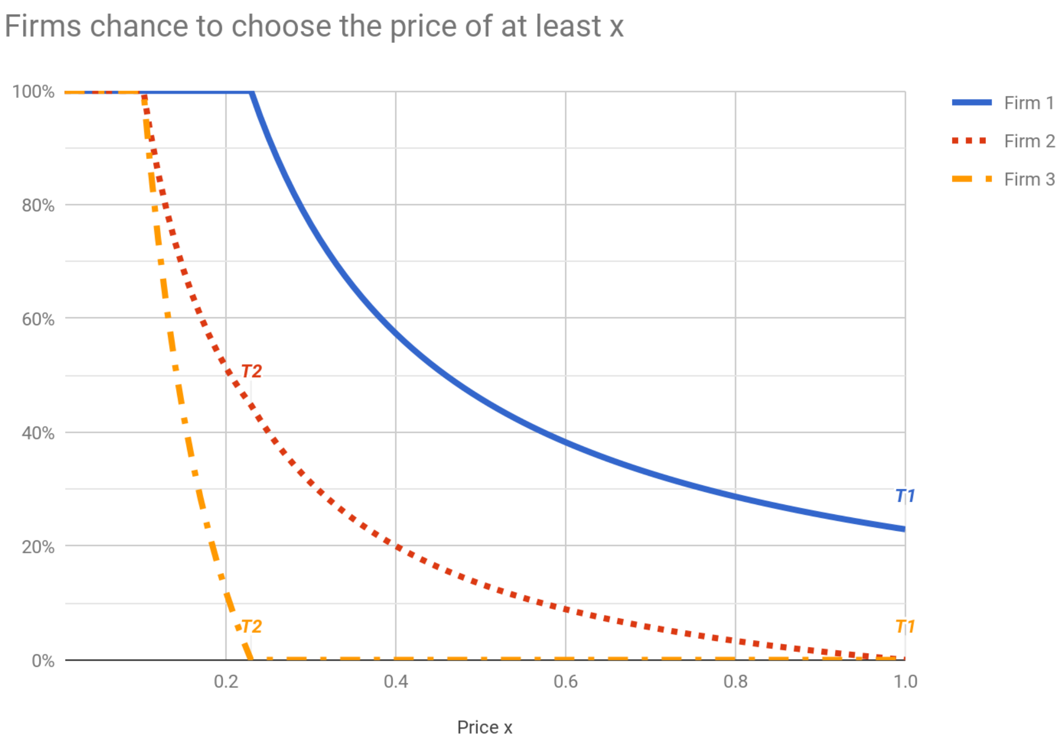

Figure 4: Inverse CDF for every firm at every x, following the results summarized in Table 2. This is the expected behavior for every seller, they cannot have a better utility level following another CDF (strategy). (Reproduced from [8].)

Figure 5: Utility of every firm after applying (1) for all x. Note that, in equilibrium, the utility is the same at the interval within the firm's support. (Reproduced from [8].).

Chapter 3 Reversed Bertrand Network

This chapter introduces a way of using the existing model in a reversed manner: finding the size of the market from the prices, considering that companies are acting in equilibrium. Furthermore, in the next section, it is shown a case of a study demonstrating the application of this technique

3.1 Preliminaries

Understand the market potential is crucial for success or failure. For this work, it is proposed a different approach to analyze an existing market by looking at how companies are pricing. Considering that the Bertrand Network model is maturing in a growing literature of networked competitions, it can be said this approach is an innovative way of analyzing the competition.

The following subsections will explain a work that has been accepted by the scientific community. In the existing model, from the competition network and with the size of the markets, it is possible to find the CDF and the best pricing range for each company. In the published work [8] the opposite has been proposed: From prices, it is possible to discover the size of the markets and have a hint of how the competition looks like. The idea is to figure out what has motivated the companies to be pricing the way they are doing.

3.2 Reversed application

To demonstrate how to use it, the same warm-up example from the last section is used. Recall the results in **Section 2.3:** $t_1 = 1$, $t_2 = 0.2294520548$ and $t_3 = 0.102739726$. For the sake of simplicity, t_2 and t_3 have been rounded to 0.23 and 0.10 respectively.

From the results in [8], it is needed to collect prices to begin the analysis. Consider prices collected from day d_1 to d_n and let $P_{i,k,j}$ be the j-th price from the k-th day of the i-th firm. Let P_i be a set of non-decreasing prices of the firm i. Finally, let P be the union of every P_i.

Since the existing model use prices from 0 to 1, let m be the maximum price in P and map every price in P to be in $[0,1]$. Formally, let $f: P \rightarrow [0,1]$ as follows:

$$f(x) = \frac{x}{m}$$

Not every price dataset can be used to find the CDF of the firms. If it is believed the dataset is large enough[12], the inverse CDF can be calculated as follows:

$$\overline{F}_i(x) = \frac{|P_i| - C_i(x)}{|P_i|} \tag{2}$$

Where $C_i(x)$ is a function that returns the number of elements in P_i that is lesser than x. Note that finding the inverse CDF from the formula above is the same as saying the probability of randomly choosing the value of x or higher.

To better understand the proposed analysis, consider that prices have been collected and the following information has been extracted.

- $max(P_1) = 1000€$
- $min(P_1) = 230€$
- $max(P_2) = 940€$
- $min(P_2) = 120€$
- $max(P_3) = 200€$

[12] The more prices are collected, the more reliable the data set is. Few prices may not be enough to represent the exact curve of the desired inverse CDF function.

- $min(P_3) = 100€$

The values listed above are graphically represented in the next Figure.

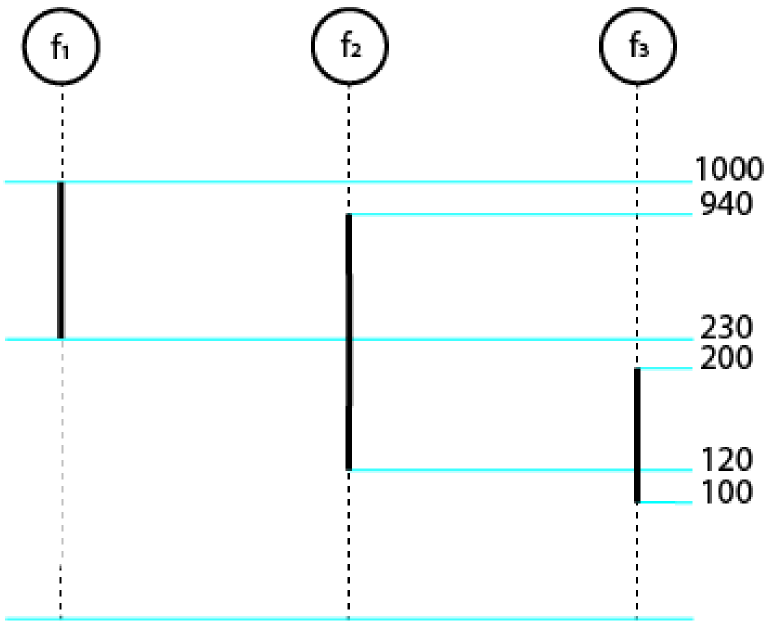

Figure 6: "Observed" pricing range of each firm.

The pricing range of each firm suggests that the competition network is a line of three sellers. It is necessary to be flexible when determining the support of each company. If $max(P_2) = 940$, it doesn't mean the company would not choose the price of 950 euros: the price could just not have been collected. Therefore, when trying to identify in which equilibrium the observed prices belongs, some adaptations may be necessary.

From the results in [7], the "observed" prices in Figure 6 seems to belong to the Mixed NE from competition with 3 sellers in line. The expected competition network is represented below.

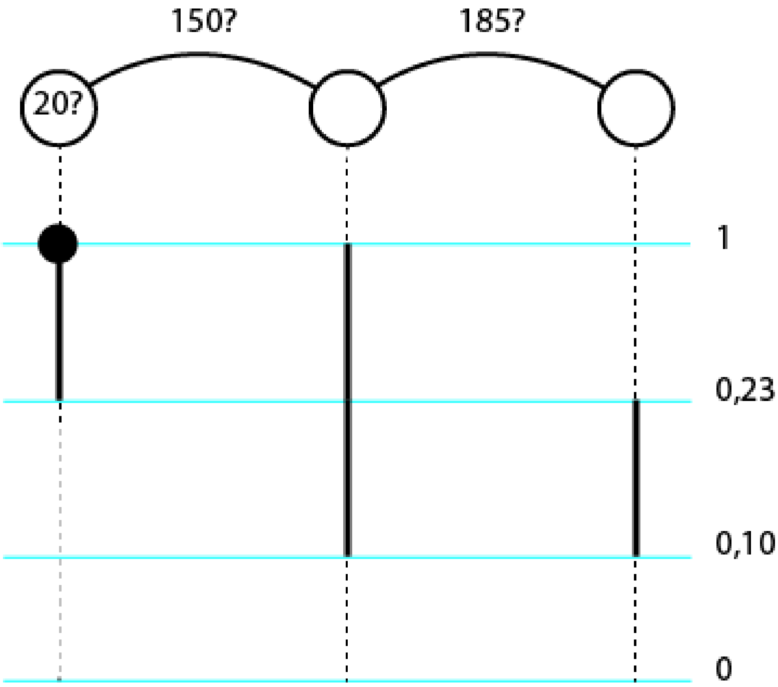

Figure 7: Expected competition. The question marks represent if it is possible to find the size of the markets by only knowing the prices. (Reproduced from [8].)

Note that $t_1 = 1$ as it would be in any equilibrium [7]. Assume that $\overline{F}_2(t_2) = 0.45$ and $\overline{F}_1(t_1) = 0.23$, values found after applying (2) in P_2 and P_1 respectively.

Applying the model in the reversed manner. By following the equalities that must be satisfied in equilibrium:

$$u_1(t_1) = u_1(t_2)$$

$$t_1 * \alpha_1 = t_2\left(\alpha_1 + \beta_{1,2}\overline{F}_2(t_2)\right)$$

$$\frac{t_2}{t_1} = \frac{\alpha_1}{\alpha_1 + \beta_{1,2}\overline{F}_2(t_2)} \tag{3}$$

$$u_2(t_1) = u_2(t_2)$$

$$t_1\left(\alpha_2 + \beta_{1,2}\overline{F}_1(t_1)\right) = t_2\left(\alpha_2 + \beta_{1,2}\right)$$

$$\frac{t_2}{t_1} = \frac{\alpha_2 + \beta_{1,2}\overline{F}_1(t_1)}{\alpha_2 + \beta_{1,2}} \tag{4}$$

$$u_2(t_2) = u_2(t_3)$$

$$t_2\left(\alpha_2 + \beta_{1,2}\right) = t_3\left(\alpha_2 + \beta_{1,2} + \beta_{2,3}\right)$$

$$\frac{t_2}{t_3} = \frac{\alpha_2 + \beta_{1,2}}{\alpha_2 + \beta_{1,2} + \beta_{2,3}} \tag{5}$$

$$u_3(t_2) = u_3(t_3)$$

$$t_2\left(\alpha_3 + \beta_{2,3}\overline{F}_2(t_2)\right) = t_3\left(\alpha_3 + \beta_{2,3}\right)$$

$$\frac{t_3}{t_2} = \frac{\alpha_3 + \beta_{2,3}\overline{F}_2(t_2)}{\alpha_3 + \beta_{2,3}} \tag{6}$$

Instead of finding the exact size of the markets, the following is used to reduce the problem to find just the ratio of the markets:

$$\alpha_1 = a * \beta_{1,2}$$
$$\alpha_2 = b * \beta_{1,2}$$
$$\alpha_3 = c * \beta_{1,2}$$
$$\beta_{2,3} = k * \beta_{1,2}$$

The market share $\beta_{1,2}$ (chosen by convenience) is always positive, otherwise, the market share would not exist.

From (3):

$$t_2 = \frac{a * \beta_{1,2}}{a * \beta_{1,2} + \beta_{1,2}\overline{F}_2(t_2)}$$

$$t_2 = \frac{a}{a + \overline{F}_2(t_2)}$$

$$a = \frac{t_2\overline{F}_2(t_2)}{1 - t_2}$$

Since “*a*” is defined only by constant values, then it is possible to treat “*a*” as a constant value. Proceeding with (4):

$$t_2 = \frac{b * \beta_{1,2} + \beta_{1,2} * \overline{F}_1(t_1)}{b * \beta_{1,2} + \beta_{1,2}}$$

$$t_2 = \frac{b + \overline{F}_1(t_1)}{b + 1}$$

$$b = \frac{\overline{F}_1(t_1) - t_2}{t_2 - 1}$$

From (5):

$$\frac{t_2}{t_3} = 1 + \frac{\beta_{2,3}}{\alpha_2 + \beta_{1,2}}$$

$$\frac{t_2}{t_3} = 1 + \frac{k}{b + 1}$$

$$k = \left(\frac{t_2}{t_3} - 1\right)(b + 1)$$

From (6):

$$\frac{t_3}{t_2} = \frac{c + k\overline{F}_2(t_2)}{c + k}$$

$$\frac{t_3}{t_2} * c + \frac{t_3}{t_2} * k - c = k * \overline{F}_2(t_2)$$

$$c = k\frac{\overline{F}_2(t_2) - \frac{t_3}{t_2}}{\frac{t_3}{t_2} - 1}$$

This concludes the calculation and now it is possible to find the size of each market share by setting a value for $\beta_{1,2}$. Note that there are many possible values for $\beta_{1,2}$ and each will lead to different utility levels for each firm, the since the results are based on the ratio of each other market over $\beta_{1,2}$, the prices and CDF will be the same.

If $\beta_{1,2} = 150$ the following holds:

$$\alpha_1 = 20.16233766$$

$$\alpha_2 = 0$$
$$\alpha_3 = 0.0455$$
$$\beta_{2,3} = 195$$

It is worth noting that the prices have been rounded earlier, thus the approximated values for the markets.

3.3 Properties

First and foremost, this section presents profs and claims about the possibility of modifying the competition network to justify the observed prices. In other words, it just states that specific modifications (like adding a new seller to the network) will converge the prices and CDFs to be closer to the observed prices, but it does not guarantee the results will represent a NE. It is always necessary to verify the equilibrium in the modified network. The set of pricing ranges and strategies can be tested if they represents a NE in polynomial time by testing a Linear Program with 8 constraints [7], which basically tests if the utility level of every seller is the same when pricing at the bounds of their support, without having to test every price x.

It is important to understand whether the analyst has a dataset with enough prices that can be used or not. Next, Theorem 3.1 and its corollaries will be useful to enhance the results of the application of the reversed method. Furthermore, it is also provided a way to test if the equilibrium form of a chosen competition network for a case of study could have given the observed prices.

Theorem 3.1. *Every inverse CDF is upper bounded.*

Proof. Fix any network and any equilibrium. Label firms such as $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$. Let $\alpha_i \geq \alpha_j$, $1 \leq i < j \leq n$ with $x_1, x_2 \in S_i$ and $x_3, x_4 \in S_j$, $x_1 \geq x_2$ and $x_3 \geq x_4$, the following must hold:

$$u_i(x_2) = u_i(x_1)$$

$$x_2\left(\alpha_i + \sum_{k \in N(i)} \beta_{i,k}\overline{F}_k(x_2)\right) = x_1\left(\alpha_i + \sum_{k \in N(i)} \beta_{i,k}\overline{F}_k(x_1)\right)$$

$$\alpha_i = \frac{x_1 \sum_{k\in N(i)} \beta_{i,k}\overline{F}_k(x_1) - x_2 \sum_{k\in N(i)} \beta_{i,k}\overline{F}_k(x_2)}{x_2 - x_1}$$

$$a_i \geq a_j$$

$$\frac{x_1 \sum_{k\in N(i)} \beta_{i,k}\overline{F}_k(x_1) - x_2 \sum_{k\in N(i)} \beta_{i,k}\overline{F}_k(x_2)}{x_2 - x_1} \geq \frac{x_3 \sum_{k\in N(i)} \beta_{j,k}\overline{F}_k(x_3) - x_4 \sum_{k\in N(i)} \beta_{j,k}\overline{F}_k(x_4)}{x_4 - x_3}$$

$$x_1 \sum_{k\in N(i)} \beta_{i,k}\overline{F}_k(x_1) - x_2 \sum_{k\in N(i)} \beta_{i,k}\overline{F}_k(x_2) \leq \frac{x_2 - x_1}{x_4 - x_3}\left(x_3 \sum_{k\in N(i)} \beta_{j,k}\overline{F}_k(x_3) - x_4 \sum_{\mathrm{k}\in N}(i)\beta_{j,k}\overline{F}_k(x_4)\right)$$

Next, any $\overline{F}_m(x)$, $m \in N(i)$ has an upper bound which must be respected so the above inequality is true. Considering $N(i,-m)$ is the set of neighbors of f_i excluding m:

$$x\left(\alpha_i + \sum_{k\in N(i)} \beta_{i,k}\overline{F}_k(x)\right) = x\left(\alpha_i + \beta_{i,m}\overline{F}_m(x) + \sum_{k\in N(i,-m)} \beta_{i,k}\overline{F}_k(x)\right)$$

Therefore:

$$\begin{aligned} &x_1\beta_{i,m}\overline{F}_m(x_1) - x_2\beta_{i,m}\overline{F}_m(x_2) \\ &\quad + x_1 \sum_{k\in N(i,-m)} \beta_{i,k}\overline{F}_k(x_1) \\ &\quad - x_2 \sum_{k\in N(i,-m)} \beta_{i,k}\overline{F}_k(x_2) \\ &\quad \leq \frac{x_2 - x_1}{x_4 - x_3}\left(\begin{array}{c} x_3\beta_{j,m}\overline{F}_m(x_3) - x_4\beta_{j,m}\overline{F}_m(x_4) \\ + x_3 \sum_{k\in N(j,-m)} \beta_{j,k}\overline{F}_k(x_3) - x_4 \sum_{k\in N(j,-m)} \beta_{j,k}\overline{F}_k(x_4) \end{array}\right) \end{aligned}$$

By moving every possible $\overline{F}_m(x_i)$ to the left side of the inequality:

$$\begin{aligned} &x_1\beta_{i,m}\overline{F}_m(x_1) - x_2\beta_{i,m}\overline{F}_m(x_2) \\ &\quad - \frac{x_2 - x_1}{x_4 - x_3}\begin{pmatrix} x_3\beta_{j,m}\overline{F}_m(x_3) \\ -x_4\beta_{j,m}\overline{F}_m(x_4) \end{pmatrix} \\ &\quad \leq \frac{x_2 - x_1}{x_4 - x_3}\begin{pmatrix} x_3 \sum_{k\in N(j,-m)} \beta_{j,k}\overline{F}_k(x_3) \\ -x_4 \sum_{\mathrm{k}\in N(j,-m)} \beta_{j,k}\overline{F}_k(x_4) \end{pmatrix} \\ &\quad - x_1 \sum_{k\in N(i,-m)} \beta_{i,k}\overline{F}_k(x_1) \end{aligned}$$

$$+x_2 \sum_{k \in N(i,-m)} \beta_{i,k} \overline{F}_k(x_2) \tag{7}$$

It is worth noting that $\beta_{j,m}$ exists if, and only if there is a competition between f_j and f_m. The expression above shows that any $\overline{F}_m(x)$, for $m \in N(i)$ is upper bounded. Note that it bounds every captive market $i < n$. The last captive market is bounded in the following condition:

$$\alpha_n \geq 0$$

$$\alpha_n = \frac{x_1 \sum_{k \in N(n)} \beta_{n,k} \overline{F}_k(x_1) - x_2 \sum_{k \in N(n)} \beta_{n,k} \overline{F}_k(x_2)}{x_2 - x_1} \geq 0$$

$$x_1 \sum_{k \in N(n)} \beta_{n,k} \overline{F}_k(x_1) - x_2 \sum_{k \in N(n)} \beta_{n,k} \overline{F}_k(x_2) \leq 0$$

$$x_1 \sum_{k \in N(n)} \beta_{i,k} \overline{F}_k(x_1) \leq x_2 \sum_{k \in N(n)} \beta_{n,k} \overline{F}_k(x_2)$$

$$\sum_{k \in N(n)} \beta_{n,k} \overline{F}_k(x_1) \leq \frac{x_2}{x_1} \sum_{k \in N(n)} \beta_{n,k} \overline{F}_k(x_2)$$

For $\overline{F}_m(x_1)$ for any seller m:

$$\beta_{n,\mathrm{m}} \overline{F}_m(x_1) + \sum_{k \in N(n,-m)} \beta_{n,k} \overline{F}_k(x_1) \leq \frac{x_2}{x_1} \sum_{k \in N(n)} \beta_{n,k} \overline{F}_k(x_2)$$

$$\beta_{n,\mathrm{m}} \overline{F}_m(x_1) \leq \frac{x_2}{x_1} \sum_{k \in N(n)} \beta_{n,k} \overline{F}_k(x_2) - \sum_{k \in N(n,-m)} \beta_{n,k} \overline{F}_k(x_1)$$

$$\overline{F}_m(x_1) \leq \frac{\frac{x_2}{x_1} \sum_{k \in N(n)} \beta_{n,k} \overline{F}_k(x_2) - \sum_{k \in N(n,-m)} \beta_{n,k} \overline{F}_k(x_1)}{\beta_{n,\mathrm{m}}} \tag{8}$$

Or even:

$$\overline{F}_m(x_1) - \frac{x_2}{x_1} \overline{F}_m(x_2) \leq \frac{\frac{x_2}{x_1} \sum_{k \in N(n,-\mathrm{m})} \beta_{n,k} \overline{F}_k(x_2) - \sum_{k \in N(n,-m)} \beta_{n,k} \overline{F}_k(x_1)}{\beta_{n,\mathrm{m}}} \tag{9}$$

Which holds for all $x_1, x_2 \in S_n, x_1 \geq x_2$. Therefore, from (7) and (9), every firm i has CDFs in his utility in which they have an upper bound condition that must be respected so $\alpha_n \geq 0$ and $\alpha_i \geq \alpha_j$ holds.

■

The next corollaries follow from **Theorem 3.1**.

Corollary 3.1. *Any dataset is called reliable if, for every inverse CDF function calculated from the dataset using (2), its upper bound condition from Theorem 1 is true and every upper bound is lesser or equal than 1. Furthermore, a non-reliable dataset suggests hidden sellers exist in the network or the price data set is too small to be trusted.*

This claims just formalize the idea that any calculated CDF must respect the upper bound from the **Theorem 4.1**.

Consider any equilibrium with n sellers and a set of prices $T = \{t_1, \dots, t_k\}$, where $1 = t_1 > \cdots > t_k > 0$. After calculating the CDF for every firm i for every price t_j in the system using (2), the dataset can be called reliable if, $\overline{F}_i(t_j)$ for every i and j, the inequality in (7) and (9) is true and every upper bound condition is lesser or equal than 1.

If this is the case, every condition like $\alpha_i \geq \alpha_j \geq 0$ from the equilibrium sketch will be respected.

■

Corollary 3.2. *If an upper bound condition from Theorem 3.1 is greater than 1, there may exist hidden sellers in the network that justify the prices companies are using.*

The idea of adding sellers to the network comes from the fact that inequalities must be true in any network. If any is false, then sellers might be missing in the network. So, if the missing sellers were considered in the calculations, the same inequality would be true.

Consider the case where $\overline{F}_m(x_1)$ wants to be upper bounded. From (7):

$$x_1 \beta_{i,m} \overline{F}_m(x_1) \leq \frac{x_2 - x_1}{x_4 - x_3} \begin{pmatrix} x_3 \sum\limits_{k \in N(j)} \beta_{j,k} \overline{F}_k(x_3) \\ -x_4 \sum\limits_{k \in N(j)} \beta_{j,k} \overline{F}_k(x_4) \end{pmatrix} - x_1 \sum_{k \in N(i,-m)} \beta_{i,k} \overline{F}_k(x_1) + x_2 \sum_{k \in N(i)} \beta_{i,k} \overline{F}_k(x_2)$$

$$\overline{F}_m(x_1) \leq \frac{\frac{x_2 - x_1}{x_4 - x_3} \begin{pmatrix} x_3 \sum_{k \in N(j)} \beta_{j,k} \overline{F}_k(x_3) \\ -x_4 \sum_{k \in N(j)} \beta_{j,k} \overline{F}_k(x_4) \end{pmatrix} - x_1 \sum_{k \in N(i,-m)} \beta_{i,k} \overline{F}_k(x_1) + x_2 \sum_{k \in N(i)} \beta_{i,k} \overline{F}_k(x_2)}{x_1 \beta_{i,m}} \leq 1$$

If the above condition is false, it is possible to add sellers to the network, so the inverse CDF function will be lesser or equal than 1. The following must hold so the above condition is true:

$$\frac{x_2 - x_1}{x_4 - x_3}\left(\begin{array}{c} x_3 \sum_{k \in N(j)} \beta_{j,k}\overline{F}_k(x_3) \\ -x_4 \sum_{k \in N(j)} \beta_{j,k}\overline{F}_k(x_4) \end{array}\right) - x_1 \sum_{k \in N(i,-m)} \beta_{i,k}\overline{F}_k(x_1) + x_2 \sum_{k \in N(i)} \beta_{i,k}\overline{F}_k(x_2) \leq x_1\beta_{i,m}$$

The overall result of the left part of the inequality will get smaller as more sellers are added to the network. It is enough to prove that any seller s added to the network as a neighbor of j, containing at least the same support as f_j, decreases the following result:

$$x_3 \sum_{k \in N(j)} \beta_{j,k}\overline{F}_k(x_3) - x_4 \sum_{k \in N(j)} \beta_{j,k}\overline{F}_k(x_4)$$

Therefore, considering $N'(j) = N(j) \cup \{s\}$, a function that unions a seller s and the set of neighbors of j:

$$x_3 \sum_{k \in N\prime(j)} \beta_{j,k}\overline{F}_k(x_3) - x_4 \sum_{k \in N\prime(j)} \beta_{j,k}\overline{F}_k(x_4) \leq x_3 \sum_{k \in N(j)} \beta_{j,k}\overline{F}_k(x_3) - x_4 \sum_{k \in N(j)} \beta_{j,k}\overline{F}_k(x_4)$$

$$x_3 \sum_{k \in N\prime(j,-s)} \beta_{i,k}\overline{F}_k(x_3) - x_4 \sum_{k \in N\prime(j,-s)} \beta_{i,k}\overline{F}_k(x_4) + x_3\beta_{i,s}\overline{F}_s(x_3) - x_4\beta_{i,s}\overline{F}_s(x_4)$$
$$\leq x_3 \sum_{k \in N(j)} \beta_{i,k}\overline{F}_k(x_3) - x_4 \sum_{k \in N(j)} \beta_{i,k}\overline{F}_k(x_4)$$

Note that $N'(j,-s)$ represents the neighbors of j after the seller s has been added to the network, whereas $N(j)$ represents the set of neighbors of j where the seller s didn't exist. In other words, $N'(j,-s) = N(j)$.

$$x_3\beta_{i,s}\overline{F}_s(x_3) - x_4\beta_{i,s}\overline{F}_s(x_4) \leq 0$$

$$x_3\beta_{i,s}\overline{F}_s(x_3) \leq x_4\beta_{i,s}\overline{F}_s(x_4)$$

$$x_3\overline{F}_s(x_3) \leq x_4\overline{F}_s(x_4)$$

$$x_4\overline{F}_s(x_3) \leq x_3\overline{F}_s(x_3) \leq x_4\overline{F}_s(x_4)$$

$$\overline{F}_s(x_3) \leq \overline{F}_s(x_4)$$

If $x_3 > x_4$ and $x_3, x_4 \in S_s$, the last inequality is strict. Note that this result does not depend on $\beta_{i,s}$. The last inequality does respect the fact that $\overline{F}_s(x)$ may decrease as x increases. This concludes the prof, and if $\overline{F}_m(x_1) > 1$ then it may suggest that more companies are competing in the network since many sellers can be added until the condition $\overline{F}_m(x_1) \leq 1$ is true.

■

Corollary 3.3. *If an upper bound condition (9) from Theorem 4.1 is false, there may exist hidden sellers in the network that justify the prices companies are using*

Equation (9) states a condition which must be respected so $\alpha_i \geq 0$ is true. If this is not the case, then the following must be true:

$$\overline{F}_m(x_1) - \frac{x_2}{x_1}\overline{F}_m(x_2) \leq \frac{\frac{x_2}{x_1}\sum_{k\in N(i,-\mathrm{m})} \beta_{i,k}\overline{F}_k(x_2) - \sum_{k\in N(i,-m)} \beta_{i,k}\overline{F}_k(x_1)}{\beta_{i,\mathrm{m}}} + \varepsilon$$

Where ε is a value that must be added so the above inequality will be true. Next, it is shown that it is possible to increase the right side of the inequality by just adding a seller s to the network in a similar way done in **Corollary 4.2**. Therefore, recalling that $N'(i) = N(i) \cup \{s\}$, it must be true that:

$$\frac{\frac{x_2}{x_1}\sum_{k\in N(i,-\mathrm{m})} \beta_{i,k}\overline{F}_k(x_2) - \sum_{k\in N(i,-m)} \beta_{i,k}\overline{F}_k(x_1)}{\beta_{i,\mathrm{m}}} \leq \frac{\frac{x_2}{x_1}\sum_{k\in N'(i-\mathrm{m})} \beta_{i,k}\overline{F}_k(x_2) - \sum_{k\in N'(i,-m)} \beta_{i,k}\overline{F}_k(x_1)}{\beta_{i,\mathrm{m}}}$$

$$\frac{x_2}{x_1}\sum_{k\in N(i,-m)} \beta_{i,k}\overline{F}_k(x_2) - \sum_{k\in N(i,-m)} \beta_{i,k}\overline{F}_k(x_1) \leq \frac{x_2}{x_1}\sum_{k\in N'(i,-m)} \beta_{i,k}\overline{F}_k(x_2) - \sum_{k\in N'(i,-m)} \beta_{i,k}\overline{F}_k(x_1)$$

$$\frac{x_2}{x_1}\sum_{k\in N(i,-m)} \beta_{i,k}\overline{F}_k(x_2) - \sum_{k\in N(i,-m)} \beta_{i,k}\overline{F}_k(x_1)$$
$$\leq \frac{x_2}{x_1}\sum_{k\in N'(i,-m,-s)} \beta_{i,k}\overline{F}_k(x_2) - \sum_{k\in N'(i,-m,-s)} \beta_{i,k}\overline{F}_k(x_1) + \frac{x_2}{x_1}\beta_{i,s}\overline{F}_s(x_2) - \beta_{i,s}\overline{F}_s(x_1)$$

$$0 \leq \frac{x_2}{x_1}\beta_{i,s}\overline{F}_s(x_2) - \beta_{i,s}\overline{F}_s(x_1)$$

$$0 \leq \frac{x_2}{x_1}\beta_{i,s}\overline{F}_s(x_2) - \beta_{i,s}\overline{F}_s(x_1)$$

$$\beta_{i,s}\overline{F}_s(x_1) \leq \frac{x_2}{x_1}\beta_{i,s}\overline{F}_s(x_2)$$

$$x_1\overline{F}_s(x_1) \leq x_2\overline{F}_s(x_2)$$

$$x_2\overline{F}_s(x_1) \leq x_1\overline{F}_s(x_1) \leq x_2\overline{F}_s(x_2)$$

$$x_2\overline{F}_s(x_1) \leq x_2\overline{F}_s(x_2)$$

$$\overline{F}_s(x_1) \leq \overline{F}_s(x_2)\ \forall \mathrm{x}_1, \mathrm{x}_2 \in [0,1], \mathrm{x}_1 \geq \mathrm{x}_2$$

If $x_1 > x_2$, the above inequality is strict. This is always true, therefore, any seller s can be added to the network since it implies $\varepsilon > 0$. So, if it is the case that (9) is false, it may be true that sellers are missing in the network that could have made (9) true.

■

Next observation provides a claim whether the chosen network is correct or not.

Observation 3.1. *Fix any network and any collected prices. It is possible to verify if there exists any size of markets that satisfy the observed prices by testing a non-linear system. Moreover, if the equation system does not have a solution, the network must be changed.*

Fix any network with n vertices and label them such as $\alpha_i \geq \alpha_{i+1}$. Call B_i the decreasing set of the boundary points in the support of f_i. Let $T = \cup_i^n B_i$ be the union of all these sets. If the fixed network is valid for the observed prices, every collected value must belong to the support of its firm. Therefore, any observed price p of the seller i must be in $\left[t_{i,b+1}, t_{i,b}\right]$, $t_{i,b} \in B_i$ for some $1 \leq b < |B_i|$.

Recall that $u_i\left(t_{i,b+1}\right) = u_i\left(t_{i,b}\right)$ since both prices belongs to S_i. The price $t_{i,b+1}$ can be found as:

$$t_{i,b+1} \cdot \left(\alpha_i + \sum_{j \in N(i)} \beta_{i,j}\overline{F}_i\left(t_{i,b+1}\right)\right) = t_{i,b} \cdot \left(\alpha_i + \sum_{j \in N(i)} \beta_{i,j}\overline{F}_i\left(t_{i,b}\right)\right)$$

$$t_{i,b+1} = \frac{t_{i,b} \cdot \left(\alpha_i + \sum_{j \in N(i)} \beta_{i,j}\overline{F}_i\left(t_{i,b}\right)\right)}{\alpha_i + \sum_{j \in N(i)} \beta_{i,j}\overline{F}_i\left(t_{i,b+1}\right)}$$

Every observed price must be inside of some interval in the support of i. Without loss of generality, assume the cases where $|B_i| = 2$ for every i. Therefore, there is only one interval in the support of i: $\left[t_p, t_q\right]$. Take the maximum and minimum observed price for the seller i, namely, $p_{i,M}$ and $p_{i,m}$. There must exist some boundary values that embrace the observed values, that is, $t_p \leq p_{i,m} \leq p_{i,M} \leq t_q$. If there is no combination of market sizes that satisfies the given property for every seller i, them it is not possible that the fixed network, in equilibrium, could have given the observed prices.

To formalize this observation, let $k = |T|$. Therefore, prices in T is in the form $1 = t_1 > t_2 > \cdots > t_k > 0$. Note that $t_k > 0$ and $t_1 = 1$ is true in any equilibrium. Let R_j be the set of sellers that contains the price t_j in his support. The following non-linear program must be satisfied for every seller i:

$$\begin{cases} t_q = 1 \geq p_{i,M} & q = 1 \\ t_q = \dfrac{t_r \cdot \left(\alpha_h + \sum_{j \in N(h)} \beta_{h,j} \cdot \overline{F}_j(t_r)\right)}{\alpha_h + \sum_{j \in N(h)} \beta_{h,j} \cdot \overline{F}_j(t_q)} \geq p_{i,M} & q > 1 \\ t_p = \dfrac{t_q \cdot \left(\alpha_i + \sum_{j \in N(i)} \beta_{i,j} \cdot \overline{F}_j(t_q)\right)}{\alpha_i + \sum_{j \in N(i)} \beta_{i,j} \cdot \overline{F}_j(t_p)} \leq p_{i,m} & \end{cases}$$

Where $t_r > t_q$, and seller h is any seller that contains t_q and t_r in his support (in other words, the seller h belongs to R_q and R_r). This formalizes that every observed price for every seller must be within the expected support of its seller. If there is no combination of captive markets and market shares that satisfies the non-linear program above, for every seller i, then it is not possible to bound the observed prices in a way that respects the NE o the fixed competition network.

■

Observation 3.2. *It is possible to know how much buyers might be willing to pay based on the observed prices*.

It is true that every observed price p may be mapped using (2), respecting the fact prices must be in $[0,1]$. Nevertheless, using the same notion from the last observation, let $p'_{i,m} = p_{i,m} \cdot p_M$ and $p'_{i,M} = p_{i,M} \cdot p_M$ be the non-mapped prices, with p_M being the highest price collected. Next is shown that this assumption can be relaxed, and another higher value can be used. The non-linear program has been changed and can be satisfied as follows, with $p \geq p_M$:

$$
\begin{cases}
t_q = p \geq p_{i,M} & q = 1 \\
t_q = p \cdot \dfrac{t_r \cdot \left(\alpha_h + \sum_{j \in N(h)} \beta_{h,j} \cdot \overline{F}_j(t_r)\right)}{\alpha_h + \sum_{j \in N(h)} \beta_{h,j} \cdot \overline{F}_j\left(t_q\right)} \geq p'_{i,M} & q > 1 \\
t_p = p \cdot \dfrac{t_q \cdot \left(\alpha_i + \sum_{j \in N(i)} \beta_{i,j} \cdot \overline{F}_j\left(t_q\right)\right)}{\alpha_i + \sum_{j \in N(i)} \beta_{i,j} \cdot \overline{F}_j\left(t_p\right)} \leq p'_{i,m} &
\end{cases}
$$

Chapter 4 Methodology

This chapter briefly explains how the research was made and how the data for the case of study has been collected.

4.1 Research Design

The starting point of this project was with the book The Art of Strategy [2]. The main point was to first understand what Game Theory is and how it is applied to economics and other decision-making problems. After a deep understating of the topic, quick research was made in the IEEE explore library and JSTOR with some key-words: game theory, Nash equilibrium, and market expansion. After finding the articles related to the area, the Google Scholar was used to find other articles that referenced the selected papers.

This work started first with the idea to study the challenges a company would face in case of expanding into an existing market. After having found some articles from this field of study, it was noticed a mathematical model has been widely used recently to determine the behavior of the companies when competing in multiple markets at the same time. In special, the Cournot Networked model has been used in all those markets, as discussed in Section 2.2.

After studying the model described in [9], it was noticed that in the Cournot Networked version firms can only compete by supplying an amount of a certain homogeneous good to the market. Since the main idea was to work with companies that would work by asking prices, the Bertrand competition model was found to be suitable. After reading the

Cournot model description and following the references in the article, it has been found that there exists a networked version for price competition.

When the Bertrand Network model description was found, another research was made by using keywords such as Bertrand, Bertrand network, networked competition, etc. It was concluded that there were no works that used the pricing competition model and, in contrast to the Cournot Networked version, the model needed improvements and areas in which it could be applied.

Since then, instead of aiming at making an algorithm to compute nash equilibria in an existing network, the goal of the work changed: enhance the model and suggest an area of application, with hypothetic examples and a real case of study.

4.2 Case Study Strategy

The main challenge was to verify which kind of competition would be suitable for the Bertrand Networked model since its main assumption is that firms should randomize to be unpredictable. It has been found that online selling fits perfectly in this scenario [4]. It has been chosen the airfare competition because it is widely known for its dynamic pricing and there are tools available to collect prices automatically.

It was considered an airfare competition in which firms ask prices for their homogeneous service: a round-trip from Porto do Rome. Companies may have loyal buyers due to a bad experience with other companies, as well as buyers that are seeking for the lowest price possible.

In this case, companies face a tradeoff between charge a higher price to extract the maximum surplus from his loyal buyers or lower down the price to be more competitive, allowing to win a portion of the market. It fits perfectly in the basic assumptions of the existing networked pricing competition model of Bertrand.

4.3 Data Collection Method

Although there are many tools available to collect prices over time, the Google flight tracking has been used to daily collect the three cheapest flight. Since many improvements to the model were required by the time data collection was needed, instead of focusing on an excellent gathering method, a service that could give the name of the companies, their price and easily available has been chosen.

Next Figure presents how the prices look like when collected, being summarized in Table 3.

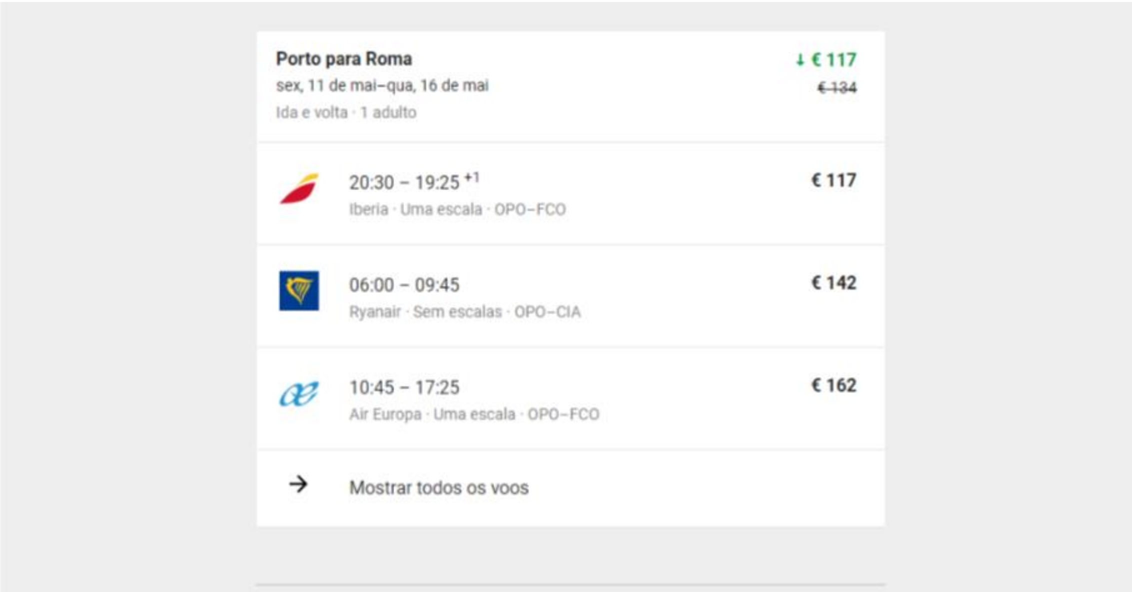

Figure 8: Image of the collection of prices

Chapter 5 Case of study

This chapter presents a case of study of the dynamic price in the airline industry. All the analysis reasoning is presented, showing also how to make use of some profs from the previous chapter.

5.1 Preliminaries

The Bertrand Network model considers that firms must mix within their support. Recalling the first section of this paper, the price of a microwave changed 9 times in response to Amazon's price changing.

Other services are empirically known as having many price changes are, but not limited to online products (as said before), bus and train tickets, accommodation services and airfares, widely known to have price changed many times in a day. In the Airline-Industry, complex strategies and methods to assign prices have been used [33], where several financial, marketing, commercial and social factors have been considered. Many techniques try to help the buyer to find the most affordable price [33, 34, 35].

In the nutshell, buyers will most benefit to buy the ticket far in advance, while sellers will do their best to figure out the demand for each flight with the huge database of previous flight and sell they possess [34].

Since there are many price changes for the same service (flight from A to B), air transport service has been chosen as a case of study. It was collected prices for the same route using the Google Flight tracking report. Each report gives three cheapest prices for that day. 6 companies appeared for the route Porto–Rome 11th – 16th May (roundtrip): Ryanair, Air Europa, Brussels Airlines, TAP, Vueling, and Iberia, yielding 72 prices combined.

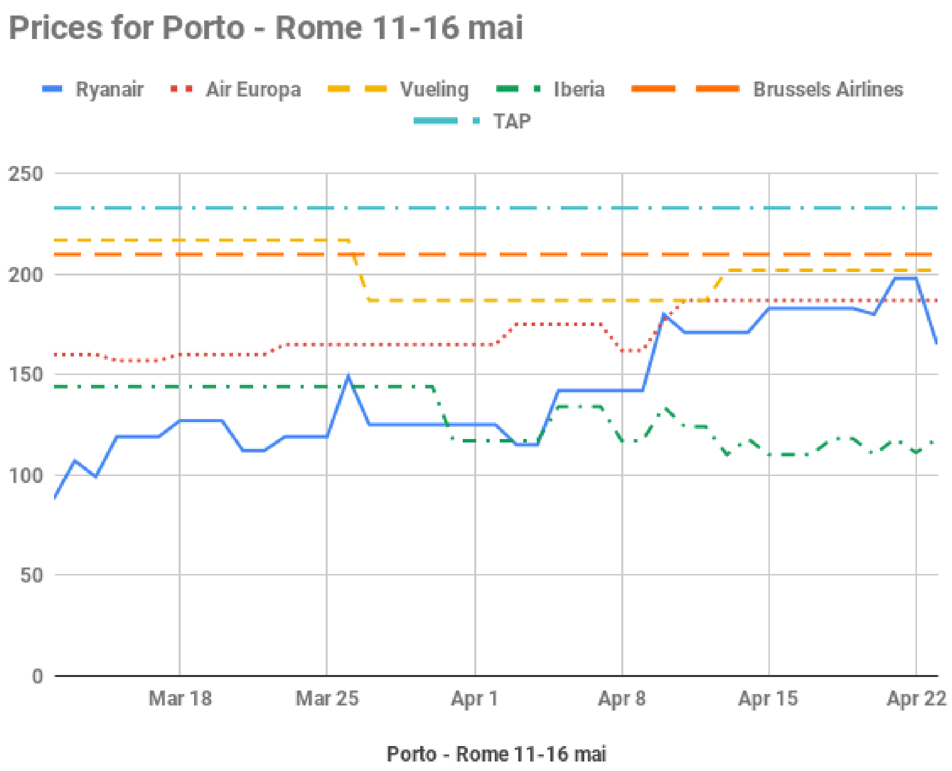

Figure 9: Lower bound for each company per day in euro, collected from March 12 to March 23.

Table 3: Price data set. Strikethrough bolded red prices represent prices that haven't appeared on the report of that day, which means that the next price of that firm in the dataset has been used as a lower bound.

	Ryanair	Air Europa	Brussels Airlines	TAP	Vueling	Iberia
Mar 12	88	160	210	**~~233~~**	**~~217~~**	**~~144~~**
Mar 13	107	160	210	**~~233~~**	**~~217~~**	**~~144~~**
Mar 14	99	160	210	**~~233~~**	**~~217~~**	**~~144~~**
Mar 15	**~~119~~**	157	210	**~~233~~**	**~~217~~**	**~~144~~**
Mar 16	**~~119~~**	**~~157~~**	**~~210~~**	**~~233~~**	**~~217~~**	**~~144~~**
Mar 17	119	**~~157~~**	**~~210~~**	**~~233~~**	**~~217~~**	**~~144~~**
Mar 18	127	160	210	**~~233~~**	**~~217~~**	**~~144~~**
Mar 19	**~~127~~**	**~~160~~**	**~~210~~**	**~~233~~**	**~~217~~**	**~~144~~**
Mar 20	**~~127~~**	**~~160~~**	**~~210~~**	**~~233~~**	**~~217~~**	**~~144~~**
Mar 21	112	160	**~~210~~**	233	**~~217~~**	**~~144~~**
Mar 22	**~~112~~**	**~~160~~**	**~~210~~**	**~~233~~**	**~~217~~**	**~~144~~**
Mar 23	119	165	**~~210~~**	**~~233~~**	217	**~~144~~**
Mar 24	**~~119~~**	**~~165~~**	**~~210~~**	**~~233~~**	**~~217~~**	**~~144~~**
Mar 25	**~~119~~**	**~~165~~**	**~~210~~**	**~~233~~**	**~~217~~**	**~~144~~**
Mar 26	149	165	**~~210~~**	**~~233~~**	**~~217~~**	144
Mar 27	125	165	**~~210~~**	**~~233~~**	187	**~~144~~**
Mar 28	**~~125~~**	**~~165~~**	**~~210~~**	**~~233~~**	**~~187~~**	**~~144~~**
Mar 29	**~~125~~**	**~~165~~**	**~~210~~**	**~~233~~**	**~~187~~**	**~~144~~**

Mar 30	~~125~~	~~165~~	~~210~~	~~233~~	~~187~~	~~144~~
Mar 31	125	165	~~210~~	~~233~~	~~187~~	117
Apr 1	125	165	~~210~~	~~233~~	187	~~117~~
Apr 2	~~125~~	~~165~~	~~210~~	~~233~~	~~187~~	~~117~~
Apr 3	115	175	~~210~~	~~233~~	187	~~117~~
Apr 4	~~115~~	~~175~~	~~210~~	~~233~~	~~187~~	~~117~~
Apr 5	142	175	~~210~~	~~233~~	~~187~~	134
Apr 6	~~142~~	~~175~~	~~210~~	~~233~~	~~187~~	~~134~~
Apr 7	~~142~~	~~175~~	~~210~~	~~233~~	~~187~~	~~134~~
Apr 8	142	162	~~210~~	~~233~~	~~187~~	117
Apr 9	~~142~~	~~162~~	~~210~~	~~233~~	~~187~~	~~117~~
Apr 10	180	177	~~210~~	~~233~~	~~187~~	134
Apr 11	171	187	~~210~~	~~233~~	~~187~~	124
Apr 12	~~171~~	~~187~~	~~210~~	~~233~~	~~187~~	~~124~~
Apr 13	171	~~187~~	~~210~~	~~233~~	202	110
Apr 14	171	187	~~210~~	~~233~~	~~202~~	118
Apr 15	183	187	~~210~~	~~233~~	~~202~~	110
Apr 16	~~183~~	~~187~~	~~210~~	~~233~~	~~202~~	~~110~~
Apr 17	~~183~~	~~187~~	~~210~~	~~233~~	~~202~~	~~110~~
Apr 18	183	187	~~210~~	~~233~~	~~202~~	118
Apr 19	~~183~~	~~187~~	~~210~~	~~233~~	~~202~~	~~118~~
Apr 20	180	187	~~210~~	~~233~~	~~202~~	110
Apr 21	198	~~187~~	~~210~~	~~233~~	~~202~~	118
Apr 22	~~198~~	187	~~210~~	~~233~~	~~202~~	111
Apr 23	165	~~187~~	~~210~~	~~233~~	~~202~~	118
Max	198	187	210	233	217	144
Min	88	157	210	233	187	10

5.2 Guessing the competition network

To guess the competition network based on the pricing range of each company, pricing range of the sellers were drawn in a paper and by checking the results in [7] it has been concluded the competition network looked like the one shown in Figure 10 and Figure 11 it is shown the comparison of the collected price ranges and the expected competition based on what has been observed.

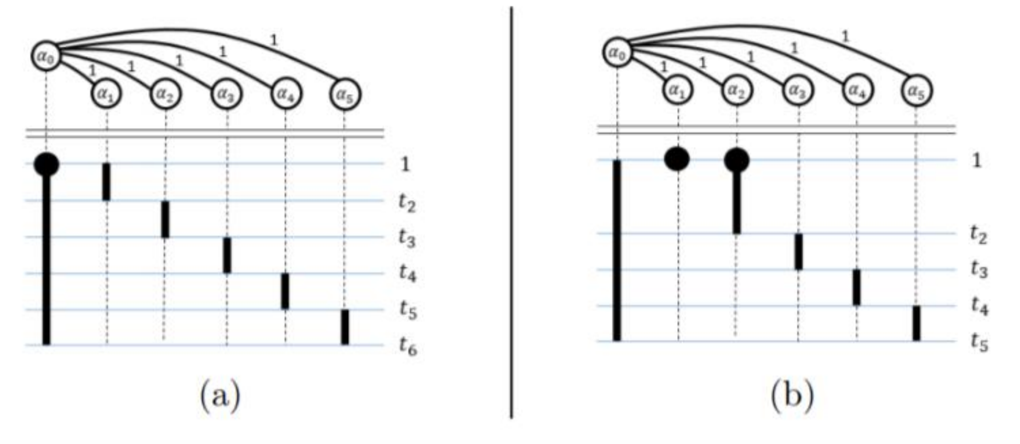

Figure 10: Typical equilibrium sketch for a unit shared market. (Reproduced from [7].)

Because TAP had only asked one price, it was assumed they are only asking the monopolized price because the market share does not worth it: they would always have a lower utility level if they try to extract any surplus form $\beta_{0,5}$. Considering that other pricing ranges are getting lower and f_5 (Ryanair) have a wide support, it was considered the observed competition looks like an instance of the right-most star network in Figure 10. Therefore, the support of each firm has been adapted and the changes can be seen in Figure 11.

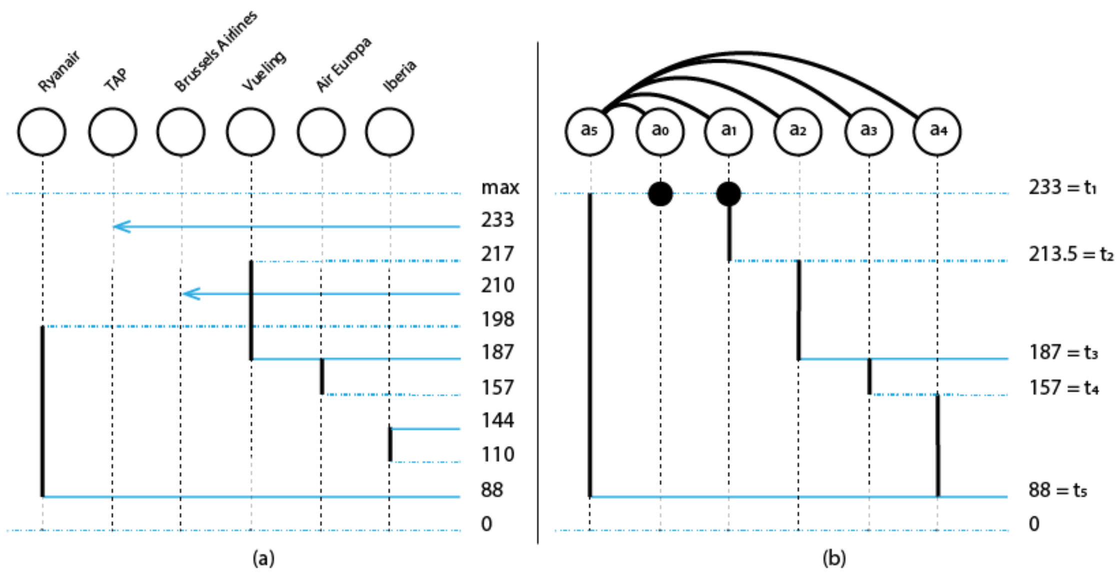

Figure 11: Observed prices (a) and adapted support of each firm (b). Each company from left to right in (a) has been labeled as: f_5, f_0, f_1, f_2, f_3 and f_4 respectively. In (a), left arrowed blue lines indicates that that company had the same maximum and minimum price in the dataset.

In any star network with the unit shared market, firm supports do not overlap. Therefore, the minimum price of Brussels Airlines f_1 and the maximum price of Vueling $f_2$has been changed to its middle point $\frac{210+217}{2} = 213.5$. Minimum and maximum pricing range of Iberia f_4 has been changed to match the minimum price of Ryanair f_5 and the minimum price of Air Europa f_3 respectively. Since the minimum price appeared in the dataset for f_1 is equal to its maximum price, the upper bound of his support has changed to match the settings of the known equilibrium in Figure 10 (b). It makes sense in the meaning of the maximum price of f_1 may not be the real maximum price the firm may ask. Raising his upper bound to 233 euros match the equilibrium requirements.

5.3 Equilibrium form

Since the equilibrium sketch for any star network considers a unit shared market, every market share $\beta_{i,j}$ may be represented simply as $\beta > 0$. Following the results in [7], firms have been labeled so that the following holds: $\alpha_i \geq \alpha_{i+1}$ for $0 \leq i \leq 4$.

In this context, since a range of prices have been observed, at least one captive market must be greater than zero. From the results in the original model [7], if there are firms mixing in equilibrium, at least one of them has a positive captive market, otherwise every seller will as a fixed price equals to his marginal cost (a pure NE).

First, prices must be in $[0,1]$. Therefore, by dividing every price by 233:

$$t_1 = t_0 = 1$$
$$t_2 = 0.91630901287$$
$$t_3 = 0.80257510729$$
$$t_4 = 0.67381974248$$
$$t_5 = 0.37768240343$$

For the sake of simplicity, since $t_0 = t_1$, t_0 has been removed from the settings. Formally, the following must hold for this competition:

$$t_1\alpha_0 = u_0(t_1)$$
$$t_2\left(\alpha_1 + \beta\overline{F}_5(t_2)\right) = u_1(t_2) = u_1(t_1) = t_1\alpha_1$$

$$t_3\left(\alpha_2+\beta\overline{F}_5(t_3)\right)=u_2(t_3)=u_2(t_2)=t_2\left(\alpha_2+\beta\overline{F}_5(t_2)\right)$$
$$t_4\left(\alpha_3+\beta\overline{F}_5(t_4)\right)=u_3(t_4)=u_3(t_3)=t_3\left(\alpha_3+\beta\overline{F}_5(t_3)\right)$$
$$t_5\left(\alpha_4+\beta\overline{F}_5(t_5)\right)=u_4(t_5)=u_4(t_4)=t_4\left(\alpha_4+\beta\overline{F}_5(t_4)\right)$$

$$u_5(t_1)=u_5(t_2)=u_5(t_3)=u_5(t_4)=u_5(t_5)$$

$$1=\overline{F}_0(t_5)=\overline{F}_0(t_4)=\overline{F}_0(t_3)=\overline{F}_0(t_2)=\overline{F}_0(t_1)$$
$$1=\overline{F}_5(t_5)>\overline{F}_5(t_4)>\overline{F}_5(t_3)>\overline{F}_5(t_2)>\overline{F}_5(t_1)=0$$
$$1=\overline{F}_4(t_5)>\overline{F}_4(t_4)=\overline{F}_4(t_3)=\overline{F}_4(t_2)=\overline{F}_4(t_1)=0$$
$$1=\overline{F}_3(t_5)=\overline{F}_3(t_4)>\overline{F}_3(t_3)=\overline{F}_3(t_2)=\overline{F}_3(t_1)=0$$
$$1=\overline{F}_2(t_5)=\overline{F}_2(t_4)=\overline{F}_2(t_3)>\overline{F}_2(t_2)=\overline{F}_2(t_1)=0$$
$$1=\overline{F}_1(t_5)=\overline{F}_1(t_4)=\overline{F}_1(t_3)=\overline{F}_1(t_2)>\overline{F}_1(t_1)>0$$

Finding every price t_i in the system. Next is shown the expressions which solves the equilibrium sketch for this network. First, for every price t_i in the system:

$$u_i(t_i)=u_i(t_{i+1}), 1\le i<5$$

$$t_i\left(\alpha_i+\beta\cdot\overline{F}_5(t_i)\right)=t_{i+1}\left(\alpha_i+\beta\cdot\overline{F}_5(t_{i+1})\right)$$

$$t_{i+1}=\frac{t_i\left(\alpha_i+\beta\cdot\overline{F}_5(t_i)\right)}{\left(\alpha_i+\beta\cdot\overline{F}_5(t_{i+1})\right)}$$

$$u_5(t_i)=u_5(t_{i+1}), 1<i<5$$

$$t_i(\alpha_5+i\cdot\beta)=t_{i+1}(\alpha_5+(i+1)\cdot\beta)$$

$$t_{i+1}=\frac{t_i(\alpha_5+i\cdot\beta)}{(\alpha_5+(i+1)\cdot\beta)}$$

$$u_5(t_1)=u_5(t_2)$$

$$t_1\left(\alpha_5+\beta\cdot\overline{F}_0(t_1)+\beta\cdot\overline{F}_1(t_1)\right)=t_2\left(\alpha_5+\beta\cdot\overline{F}_0(t_2)+\beta\cdot\overline{F}_1(t_2)\right)$$

$$\left(\alpha_5 + \beta + \beta \cdot \overline{F}_1(t_1)\right) = t_2(\alpha_5 + 2 \cdot \beta)$$

$$t_2 = \frac{\left(\alpha_5 + \beta + \beta \cdot \overline{F}_1(t_1)\right)}{(\alpha_5 + 2 \cdot \beta)}$$

The inverse CDF of every seller can be found using any price as a middle point:

$$\frac{t_4(\alpha_5 + 4\beta)}{\alpha_5 + 5\beta} = t_5 = \frac{t_4\left(\alpha_4 + \beta\overline{F}_5(t_4)\right)}{\alpha_4 + \beta}$$

$$\overline{F}_5(t_4) = \frac{(\alpha_5 + 4\beta)(\alpha_4 + \beta)}{(\alpha_5 + 5\beta) \cdot \beta} - \frac{\alpha_4}{\beta}$$

With similar reasoning, the rest of the inverse CDFs of seller 5 is:

$$\overline{F}_5(t_3) = \frac{(\alpha_5 + 3\beta)\left(\alpha_3 + \beta \cdot \overline{F}_5(t_4)\right)}{(\alpha_5 + 4\beta) \cdot \beta} - \frac{\alpha_3}{\beta}$$

$$\overline{F}_5(t_2) = \frac{(\alpha_5 + 2\beta)\left(\alpha_2 + \beta \cdot \overline{F}_5(t_3)\right)}{(\alpha_5 + 3\beta) \cdot \beta} - \frac{\alpha_2}{\beta}$$

And finally, by using t_2 as ma iddle point:

$$\frac{\alpha_5 + \beta + \beta \cdot \overline{F}_1(t_1)}{\alpha_5 + 2\beta} = t_2 = \frac{\alpha_1}{\alpha_1 + \beta \cdot \overline{F}_5(t_2)}$$

$$\overline{F}_1(t_1) = \frac{\alpha_1 \cdot (\alpha_5 + 2\beta)}{\left(\alpha_1 + \beta \cdot \overline{F}_5(t_2)\right) \cdot \beta} - \frac{\alpha_5 + \beta}{\beta}$$

For the sake of simplicity, every price t_i will be reduced as follows:

$$t_2 = \frac{\alpha_5 + \beta + \beta \cdot \overline{F}_1(t_1)}{\alpha_5 + 2 \cdot \beta}$$

$$t_3 = \frac{t_2 \cdot (\alpha_5 + 2 \cdot \beta)}{\alpha_5 + 3 \cdot \beta} \Rightarrow t_3 = \frac{\alpha_5 + \beta + \beta \cdot \overline{F}_1(t_1)}{\alpha_5 + 3 \cdot \beta}$$

$$t_4 = \frac{t_3 \cdot (\alpha_5 + 3 \cdot \beta)}{\alpha_5 + 4 \cdot \beta} \Rightarrow t_4 = \frac{\alpha_5 + \beta + \beta \cdot \overline{F}_1(t_1)}{\alpha_5 + 4 \cdot \beta}$$

$$t_5 = \frac{t_4 \cdot (\alpha_5 + 4 \cdot \beta)}{\alpha_5 + 5 \cdot \beta} \Rightarrow t_5 = \frac{\alpha_5 + \beta + \beta \cdot \overline{F}_1(t_1)}{\alpha_5 + 5 \cdot \beta}$$

Reducing the markets as a ratio of β. To simplify the calculations, every captive market has been modified to be a ration of β, similar to what have been done in the last chapter. Therefore, let $\alpha_i = r_i \cdot \beta$ for every i.

$$t_2 = \frac{r_5 \cdot \beta + \beta + \beta \cdot \overline{F}_1(t_1)}{r_5 \cdot \beta + 2 \cdot \beta} = \frac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 2}$$

$$t_3 = \frac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 3}$$

$$t_4 = \frac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 4}$$

$$t_5 = \frac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 5}$$

$$\overline{F}_1(t_1) = \frac{r_1 \cdot (r_5 + 2)}{r_1 + \overline{F}_5(t_2)} - r_5 - 1$$

$$\overline{F}_5(t_2) = \frac{(r_5 + 2)\left(r_2 + \overline{F}_5(t_3)\right)}{r_5 + 3} - r_2$$

$$\overline{F}_5(t_3) = \frac{(r_5 + 3)\left(r_3 + \overline{F}_5(t_4)\right)}{r_5 + 4} - r_3$$

$$\overline{F}_5(t_4) = \frac{(r_5 + 4)(r_4 + 1)}{r_5 + 5} - r_4$$

5.4 The correctness of the guessed competition network

Next, from **Corollary 3.1**, it is shown the dataset is not reliable, therefore bounds are used to determine the size of the captive market of each seller.

The dataset is not reliable. To prove this argument, it is enough to show that $\alpha_4 \geq \alpha_5$ fails to be true respecting the upper bound condition in (7).

From (7), by upper bounding $\overline{F}_5(t_4)$:

$$\alpha_4 \geq \alpha_5$$

$$\begin{aligned} t_4\beta\overline{F}_5(t_4) - t_5\beta\overline{F}_5(t_5) \\ & -\frac{t_5 - t_4}{t_5 - t_4}(0) \\ & \leq \frac{t_5 - t_4}{t_5 - t_4}\begin{pmatrix} t_4\beta\overline{F}_4(t_4) + t_4\beta\overline{F}_3(t_4) + t_4\beta\overline{F}_2(t_4) + t_4\beta\overline{F}_1(t_4) + t_4\beta\overline{F}_0(t_4) \\ -t_5\beta\overline{F}_4(t_5) - t_5\beta\overline{F}_3(t_5) - t_5\beta\overline{F}_2(t_5) - t_5\beta\overline{F}_1(t_5) - t_5\beta\overline{F}_0(t_5) \end{pmatrix} \\ & -0 \\ & +0 \end{aligned}$$

$$t_4\beta\overline{F}_5(t_4) - t_5\beta\overline{F}_5(t_5) \leq \frac{t_5 - t_4}{t_5 - t_4}(4t_4\beta - 5t_5\beta)$$

$$t_4\beta\overline{F}_5(t_4) \leq 4t_4\beta - 5t_5\beta + t_5\beta$$

$$t_4\overline{F}_5(t_4) \leq 4t_4 - 5t_5 + t_5$$

$$t_4\overline{F}_5(t_4) \leq 4t_4 - 4t_5$$

$$\overline{F}_5(t_4) \leq 4 - \frac{4t_5}{t_4}$$

Substituting the values of t_5 and t_4 yields $\overline{F}_5(t_4) \leq 1.75796178343$. From **Corollary 3.1**, the dataset is not reliable.

There is no combination of markets sizes, nor top price that buyers are willing to pay, that justify the price firms are asking. From observation 3.2 and without loss of generality., the following simplified non-linear program must be satisfied:

$$\begin{cases} m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 5} \leq 88 \\ m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 4} \geq 144 \\ m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 4} \leq 157 \\ m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 3} = 187 \\ m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 2} = 213.5 \end{cases}$$

Where $m \geq 233$ is the maximum price the buyers are willing to pay. The first and second condition asserts that there are two prices that bounds the expected support of f_4 and the third condition assert that the top price bounding 144 does not pass the minimum price observed from f_3. The last two conditions must be exact because it was seen in the dataset (see Figure 7).

From the last condition:

$$\frac{m \cdot r5 + m \cdot \left(1 + \overline{F}_1(t_1)\right)}{r_5 + 2} = 213.5$$

$$m \cdot r5 + m \cdot \left(1 + \overline{F}_1(t_1)\right) = 213.5 \cdot (r_5 + 2)$$

$$r_5 \cdot (m - 213.5) = 427 - m \cdot \left(1 + \overline{F}_1(t_1)\right)$$

$$r_5 = \frac{427 - m \cdot \left(1 + \overline{F}_1(t_1)\right)}{m - 213.5}$$

Continuing the penultimate condition:

$$\frac{m \cdot \left(r_5 + 1 + \overline{F}_1(t_1)\right)}{r_5 + 3} = 187$$

$$m \cdot \left(r_5 + 1 + \overline{F}_1(t_1)\right) = 187(r_5 + 3)$$

$$m\left(1 + \overline{F}_1(t_1)\right) + r_5(m - 187) = 561$$

$$m\left(1 + \overline{F}_1(t_1)\right) + \frac{427 - m\left(1 + \overline{F}_1(t_1)\right)}{m - 213.5} \cdot (m - 187) = 561$$

$$m\left(1 + \overline{F}_1(t_1)\right) + \frac{427 \cdot (m - 187) - m\left(1 + \overline{F}_1(t_1)\right) \cdot (m - 187)}{m - 213.5} = 561$$

$$\frac{m\left(1 + \overline{F}_1(t_1)\right)(m - 213.5) + 427 \cdot (m - 187) + m\left(1 + \overline{F}_1(t_1)\right) \cdot (187 - m)}{m - 213.5} = 561$$

$$m\left(1 + \overline{F}_1(t_1)\right)(-26.5) + 427 \cdot (m - 187) = 561 \cdot (m - 213.5)$$

$$-26.5m\left(1 + \overline{F}_1(t_1)\right) - 134m = -39924.5$$

$$-160.5m - 26.5m\overline{F}_1(t_1) = -39924.5$$

$$\overline{F}_1(t_1) = \frac{-39924.5 + 160.5m}{-26.5m} = \frac{-79849 + 321m}{-53m} = \frac{79849 - 321m}{53m}$$

The domain of every Inverse CDF is [0,1]. Note that, in this context, $0 < \overline{F}_1(t_1) < \overline{F}_1(t_2) = 1$.

$$\frac{79849 - 321m}{53m} > 0$$

$$79849 - 321m < 0$$

$$m < \frac{79849}{321} \approx 248.75$$

$$\frac{79849 - 321m}{53m} < 1$$

$$79849 - 321m < 53m$$

$$m > \frac{79849}{374} \approx 213.5$$

Since m must be greater or equal to 233, the bounds founded for m are valid. From the next constraint in the non-linear system:

$$\frac{m \cdot \left(r_5 + 1 + \overline{F}_1(t_1)\right)}{r_5 + 4} \leq 157$$

$$m \cdot \left(r_5 + 1 + \overline{F}_1(t_1)\right) \leq 157(r_5 + 4)$$

$$m\left(1 + \overline{F}_1(t_1)\right) + r_5(m - 157) \leq 628$$

$$m\left(1 + \overline{F}_1(t_1)\right) + \frac{427 - m\left(1 + \overline{F}_1(t_1)\right)}{m - 213.5} \cdot (m - 157) \leq 628$$

$$m\left(1 + \overline{F}_1(t_1)\right) + \frac{427 \cdot (m - 187) - m\left(1 + \overline{F}_1(t_1)\right) \cdot (m - 187)}{m - 213.5} \leq 628$$

$$\frac{m\left(1+\overline{F}_1(t_1)\right)(m-213.5)+427\cdot(m-157)+m\left(1+\overline{F}_1(t_1)\right)\cdot(157-m)}{m-213.5}\leq 628$$

$$m\left(1+\overline{F}_1(t_1)\right)(-56.5)+427\cdot(m-157)\leq 628\cdot(m-213.5)$$

$$-56.5m\left(1+\overline{F}_1(t_1)\right)-201m\leq -86681$$

$$-257.5m-56.5m\overline{F}_1(t_1)\leq -86681$$

$$\overline{F}_1(t_1)\geq\frac{-86681+257.5m}{-56.5m}$$

From the claim that $\overline{F}_1(t_1)$ is bounded:

$$\frac{-86681+257.5m}{-56.5m}>0$$

$$-86681+257.5m<0$$

$$m<\frac{86681}{257.5}\approx 336.62$$

$$\frac{-86681+257.5m}{-56.5m}<1$$

$$-86681+257.5m>-56.5m$$

$$m>\frac{86681}{314}\approx 276.05$$

From the last calculations, m is bounded from 276.05 to 336.62 in which will respect the fact that $\overline{F}_1(t_1)$ is in (0,1), for any positive captive markets market shares. The first bound founded for m is from 233 to 248.75, since the intervals are disjoint, it is not possible to respect both bounds at the same time. It is possible to conclude without having to check other constraints that, for this competition network, there is no combination of market sizes that can motivate f_1 to price from 187 euros to 213.5, doesn't matter how much the buyers are willing to pay.

The conclusion is that the competition network must be modified by adding sellers or it is completely different.

5.5 Modifying the network

Note that, in the non-linear system, the variables involved were r_5, m and $\overline{F}_1(t_1)$. The size of the captive markets influences in the price decision of every company, but in the non-linear system used to show the correctness of the network structure, adding a competitor against any seller excluding f_5 will only influence the result of $\overline{F}_1(t_1)$. Neither r_5 nor m depends on who other companies are sharing markets.

In fact, the non-linear system has no solution and adding any competitor against the firms f_0, f_1, f_2, f_3 and f_4 will not change the fact that $\overline{F}_1(t_1)$ is bounded in $(0,1)$. Therefore, in it is concluded that it is only possible to fix the network by adding sellers against f_5. Otherwise, the whole network must be changed.

By adding a seller f_6 against f_5, the non-linear system will look to something like:

$$\begin{cases} m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1) + r_6 \cdot \overline{F}_6(t_1)}{r_5 + 5 + r_6} \leq 88 \\ m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1) + r_6 \cdot \overline{F}_6(t_1)}{r_5 + 4 + r_6 \cdot \overline{F}_6(t_4)} \geq 144 \\ m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1) + r_6 \cdot \overline{F}_6(t_1)}{r_5 + 4 + r_6 \cdot \overline{F}_6(t_4)} \leq 157 \\ m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1) + r_6 \cdot \overline{F}_6(t_1)}{r_5 + 3 + r_6 \cdot \overline{F}_6(t_3)} = 187 \\ m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1) + r_6 \cdot \overline{F}_6(t_1)}{r_5 + 2 + r_6 \cdot \overline{F}_6(t_2)} = 213.5 \end{cases}$$

For the sake of the argument, suppose the support of seller 6 is $[t_5, t_3]$. Therefore, the equation system will be:

$$\begin{cases} m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 5 + r_6} \leq 88 \\ m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 4 + r_6 \cdot \overline{F}_6(t_4)} > 144 \\ m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 4 + r_6 \cdot \overline{F}_6(t_4)} \leq 157 \\ m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 3} = 187 \\ m \cdot \dfrac{r_5 + 1 + \overline{F}_1(t_1)}{r_5 + 2} = 213.5 \end{cases}$$

The two last constraints are the same of the equation system used in the last subsection. Therefore, r_5 and m constraints are still the same for the two last equalities:

$$213.5 \approx \frac{79849}{374} < 233 \leq m < \frac{79849}{321} \approx 248.75$$

$$r_5 = \frac{427 - m \cdot \left(1 + \overline{F}_1(t_1)\right)}{m - 213.5}$$

$$\overline{F}_1(t_1) = \frac{79849 - 321m}{53m}$$

After using the WolframAlpha[13] to solve the non-linear system above, the following results have been found:

$$0 < \overline{F}_6(t_4) \leq \frac{35912}{470215} = 0.07637357379$$

$$= \frac{4489}{8321 \cdot \overline{F}_6(t_4)} \leq r_6 \leq \frac{19729}{7632 \cdot \overline{F}_6(t_4)}$$

$$r_5 = \frac{268}{53} = 5.05660377358$$

The following values above will respect the non-linear equation system above, therefore a solution exists that respects the prices asked from the companies, but only after adding a new seller to the competition network.

5.6 Understanding the results

Since it was shown that the guessed competition network cannot have a solution without being modified, a new seller was added against the firm f_5, satisfying the price constraints. If the network is correct, the following information can be considered successfully discovered after the reversed application of the Bertrand Network model:

Buyers are willing to pay up to 248 euros. Considering the following constraint:

[13]https://www.wolframalpha.com/input/?i=m*(x%2B1%2By)%2F(x%2B5%2Bz)%3C%3D88,+m*(x%2B1%2By)%2F(x%2B4%2Bz*c)%3E144,+m*(x%2B1%2By)%2F(x%2B4%2Bz*c)%3C%3D157,+m*(x%2B1%2By)%2F(x%2B3)%3D187,+m*(x%2B1%2By)%2F(x%2B2)%3D213.5+,+y+%3E+0,+y+%3C%3D1,+m%3E%3D233,+x%3E0,+z%3E0,+c%3E0,+c%3C1

$$213.5 \approx \frac{79849}{374} < 233 \leq m < \frac{79849}{321} \approx 248.75$$

The captive market of the seller f_5 is roughly 5 times greater than the size of the market share β. From the following result:

$$r_5 = \frac{268}{53} = 5.05660377358$$

There is at least one more seller against f_5. For the sake of the argument, it has been assumed the support of the seller f_6 is $[t_5, t_3]$. The following probabilities for the seller f_6 is:

$$\overline{F}_6(t_3) = 0$$

$$0 < \overline{F}_6(t_4) \leq \frac{35912}{470215} = 0.07637357379$$

$$\overline{F}_6(t_5) = 1$$

The size of the captive market of the added seller f_6 is bounded and positive. From the condition above:

$$\frac{4489}{8321 \cdot \overline{F}_6(t_4)} \leq r_6 \leq \frac{19729}{7632 \cdot \overline{F}_6(t_4)}$$

By setting $\overline{F}_6(t_4) = 0.07637357379$:

$$7.06367924535 \leq r_6 \leq 33.8472662644$$

Every seller has a positive captive market. Since $\alpha_i \geq \alpha_{i+1}$, for $0 \leq i \leq 4$ and $\alpha_5 = r_5 \cdot \beta$ is positive, it can be concluded that every seller has a set of loyal buyers at their disposal.

The buyers may be categorized in 6 different ways. The equilibrium theory seeks to explain how things happen and not why [2]. By looking at the price of the companies and showing that, for the chosen network, another seller is needed so the observed prices may arise, it can be concluded the buyers may be categorized in 6 different way (one for each market share).

Chapter 6 Conclusions and future work

This chapter ends the project by presenting briefly what has been done so far, as well explaining the scientific contribution of this work, showing that this topic is seeking contributions. At the end of the second subsection, open problems are presented, which may guide the reader if the person gets interested to continue this work.

6.1 Discussion

This work has presented a way to analyze an existing competition among forms in which can be seen that clearly, firms influence the pricing decision of others even though they are not direct competitors. To be able to achieve the results shown in the last chapter, a lot of work has been demanded to identify the properties of the young and maturing Bertrand Network model, studying all the tiny details and then, presenting formal proofs that firms may exist if some conditions are not met.

Moreover, it is even possible to compute if an equilibrium may exist by solving a non-linear system of equations, from the results in **Observation 3.1**. Furthermore, **Observation 3.2** states that it is possible to guess the maximum price that the buyers from the given network, in equilibrium, are willing to pay, which could justify the observed prices.

Although the results of the case of study are not accurate since it was now shown all the size of all markets, bounds have been provided, showing that, from the proofs in this work, it is possible to analyze the market to gather information from the competition, having a glimpse of how the competition may look like if a seller decides to join the competition.

It is worth noting the reversed application of the existing model proposed in this work has been successfully accepted by the scientific community. The paper titled as *"A demonstration of an application of the Bertrand Network: Guessing the distribution of buyers within the market"* has been accepted and presented in the **18th Conference of the Portuguese Association of Information Systems (CAPSI 18)**, held in the Polytechnic Institute of Santarém - Portugal**.** The same article was also accepted in the **Federated Conference on Computer Science and Information Systems 2018 (FedCSIS 2018)**, held this year in Poland.

The paper was praised by the explanation of the existing method, as wells as the outstanding literature review presented in the introduction of the paper, which reinforces that the theme chosen for this work was successfully selected following the current needs in the Game Theory.

The article presented in the CAPSI has been chosen as one of the best articles of the conference, which will be published and index in the Lecture Notes of the Germany publishing company **Springer**. The editors invited the authors to produce another article following the same field of study, having been produced and named as *"Reversed application of the Bertrand Network"*, which is under analysis to be published and index in the **Association for Information Systems (AIS) Electronic Library (AISeL)**. The last paper focused to explain the idea behind the reversed application, its properties (proofs and claims) and a demonstration to better understand the general idea.

6.2 Open problems

In the results, it was not stated if the modified network has prices and probabilities that leads to equilibrium because there is no generic way to find a NE in any given network, but it does not change the general results from the reversed application since it provide claims that the competition network must be changed and it is very likely that the equilibrium may contain the calculated prices and probabilities.

It is needed to provide profs for other classes of graphs, in which may give more flexibility to the analyst to determine which competition structure could have motivated the firms to be pricing the way it was observed.

As stated before, currently, there is no algorithm to calculate the Nash Equilibrium of any network, which can be an interesting work to be done. Nevertheless, it is important to apply this model in other real cases, which can be used to test the profs and claims presented in this work, modifying or presenting more profs that can make the analysis easier.

Another interesting area of study would apply both gathering methods to collect prices in real time and the equilibrium verification, which can be used to identify why companies are pricing in one specific way, as well as identify when sellers leave or join other markets, based on drastic changes in their pricing decisions.

It is also important to consider that a competition in equilibrium may not be profitable to a firm, forcing the seller to leave the market. A way to calculate if a competition in equilibrium may lead to a firm leaving the market may be interesting to investigate. Furthermore, applying a production/service cost is also another important point to consider, a modification of the utility function which considers this cost was already studied by the authors of the existing model and it is presented in [3].

Last, but not least, it is important to expand the model to allow a mix of oligopolies (three or more sellers sharing the same market) and duopolies. Currently, the existing model does not allow this type of competition structure. An attempt to expand the model has been tried during the development of this work and the results are presented in [8].

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