Automatic Computation of Conservation Laws in the Calculus of Variations and Optimal Control*

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Abstract

We present analytic computational tools that permit us to identify, in an automatic way, conservation laws in optimal control. The central result we use is the famous Noether's theorem, a classical theory developed by Emmy Noether in 1918, in the context of the calculus of variations and mathematical physics, and which was extended recently to the more general context of optimal control. We show how a Computer Algebra System can be very helpful in finding the symmetries and corresponding conservation laws in optimal control theory, thus making useful in practice the theoretical results recently obtained in the literature. A Maple implementation is provided and several illustrative examples given.

Keywords: optimal control, calculus of variations, computer algebra, Noether's theorem, symmetries, conservation laws.

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1 Introduction

Optimal control problems are usually solved with the help of the famous Pontryagin maximum principle [17], which is a generalization of the classic Euler-Lagrange and Weierstrass necessary optimality conditions of the calculus of variations. The method of finding optimal solutions via Pontryagin's maximum principle proceeds through the following main three steps: (i) one defines the Hamiltonian of the problem; (ii) with the help of the maximality condition one tries to express the control variables with respect to state and adjoint variables; (iii) the Hamiltonian system is written in terms of state and adjoint variables only, and the solutions of this system of ordinary differential equations are sought. Steps (ii) and (iii) are, generally speaking, nontrivial, and very difficult (or even impossible) to implement in practice [21]. One way to address the problem is to find conservation laws, i.e., quantities which are preserved along the extremals of the problem. Such conservation laws can be used to simplify the problem [8, 9]. The question is then the following: how to determine these conservation laws? It turns out that the classic results of Emmy Noether [14, 15] of the calculus of variations, relating the existence of conservation laws with the existence of symmetries, can be generalized to the wider context of optimal control [4, 6, 23], reducing the problem to the one of discovering the invariance-symmetries. The difficulty resides precisely in the determination of the variational symmetries. While in Physics and Economics the question of existence of conservation laws is treated in a rather natural way, because the application itself suggest the symmetries (e.g., conservation of energy, conservation of momentum, income/health law, etc – all of them coming from very intuitive symmetries of the problem), from a strictly mathematical point of view, given a problem of optimal control, it is not obvious and not intuitive how one might derive a conservation law. Therefore, it would be of great practical use to have at our disposal computational means for the automatic identification of the symmetries of the optimal control problems [8, 9]. This is the motivation of the present work: to present a Maple package that can assist in this respect. The results extend the previous investigations of the authors, done in the classical context of the calculus of variations [7], to the more general and interesting setting of optimal control [17], where the application of symmetry and conservation laws is an area of current research [9, 25].

The use of symbolic mathematical software is becoming, in recent years, an effective tool in mathematics [18]. Computer algebra, also known as symbolic computation, is an interdisciplinary area of mathematics and computer science. Computer Algebra Systems, Maple as an example, facilitate the interplay of conventional mathematics with computers. They are, in some sense, changing the way we learn, teach, and do research in mathematics [1]. They can perform a myriad of symbolic mathematical operations, like analytic differentiation, integration of algebraic formulae, factoring polynomials, computing the complex roots of analytic functions, computing Taylor series expansions of functions, finding analytic solutions of ordinary or partial differential equations, etc. It is not a surprise that they are becoming popular in control theory and control engineering applications [16]. Here we use the Maple 9.5 system to find symmetries and corresponding conservation laws in optimal control. The paper is organized as follows. In §2 the problem of optimal control is introduced, the necessary definitions are given, and Noether's theorem is introduced. The method of computing conservation laws in optimal control is explained in §3, and the Maple package is then applied in §4 for computing symmetries and families of conservations laws to a diverse range of optimal control problems. In §5 we focus attention to the symbolic computation of conservation laws in the calculus of variations. The Maple procedures are given in §6, and we end §7 with some comments and directions of future work.

2 Conservation Laws in Optimal Control

The optimal control problem consists in the minimization of an integral functional,

$$I[\mathbf{x}(\cdot),\mathbf{u}(\cdot)] = \int_a^b L(t,\mathbf{x}(t),\mathbf{u}(t))\,\mathrm{d}t\,, \tag{1}$$

subject to a control system described by ordinary differential equations,

$$\dot{\mathbf{x}}(t) = \boldsymbol{\varphi}(t,\mathbf{x}(t),\mathbf{u}(t))\,, \tag{2}$$

together with appropriate boundary conditions. The Lagrangian $L(\cdot,\cdot,\cdot)$ is a real function, assumed to be continuously differentiable in $[a,b]\times\mathbb{R}^n\times\mathbb{R}^m$; $t\in\mathbb{R}$ the independent variable; $\mathbf{x}\colon[a,b]\to\mathbb{R}^n$ the vector of state variables; $\mathbf{u}\colon[a,b]\to\Omega\subseteq\mathbb{R}^m$, Ω an open set, the vector of controls, assumed to be piecewise continuous functions; and $\boldsymbol{\varphi}\colon[a,b]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ the velocity vector, assumed to be a continuously differentiable vector function. We propose a computational method that permits to obtain conservation laws for a given optimal control problem. Our method is based in the version of Noether's theorem established in [4] (see also [28]). To describe a systematic method to compute conservation laws, first we need to recall the standard definitions of *extremal*, and *conservation law*.

The central result in optimal control theory is the famous Pontryagin maximum principle [17], which gives a necessary optimality condition for the problems of optimal control.

Theorem 1 ((Pontryagin maximum principle)) *If* $(\mathbf{x}(\cdot),\mathbf{u}(\cdot))$ *is a solution of the optimal control problem (1)-(2), then there exists a non-zero pair* $(\psi_0,\boldsymbol{\psi}(\cdot))$*, where* $\psi_0\le 0$ *is a constant and* $\boldsymbol{\psi}(\cdot)$ *an n-vectorial piecewise* C^1*-smooth function with domain* $[a,b]$*, in such a way the quadruple* $(\mathbf{x}(\cdot),\mathbf{u}(\cdot),\psi_0,\boldsymbol{\psi}(\cdot))$ *satisfies the following conditions in almost all points t in the interval* $[a,b]$*:*

(i) *the Hamiltonian system*

$$\dot{\mathbf{x}}(t)^{\mathrm{T}} = \frac{\partial H}{\partial\boldsymbol{\psi}}(t,\mathbf{x}(t),\mathbf{u}(t),\psi_0,\boldsymbol{\psi}(t))\,, \tag{3}$$

$$\dot{\boldsymbol{\psi}}(t)^{\mathrm{T}} = -\frac{\partial H}{\partial\mathbf{x}}(t,\mathbf{x}(t),\mathbf{u}(t),\psi_0,\boldsymbol{\psi}(t))\,, \tag{4}$$

(ii) *the maximality condition*

$$H(t,\mathbf{x}(t),\mathbf{u}(t),\psi_0,\boldsymbol{\psi}(t)) = \max_{\mathbf{v}\in\Omega} H(t,\mathbf{x}(t),\mathbf{v},\psi_0,\boldsymbol{\psi}(t)), \tag{5}$$

with Hamiltonian

$$H(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) = \psi_0 L(t,\mathbf{x},\mathbf{u}) + \boldsymbol{\psi}^{\mathrm{T}} \cdot \boldsymbol{\varphi}(t,\mathbf{x},\mathbf{u}) \,. \tag{6}$$

Remark 2 *The right-hand side of equations (3) and (4), of the Hamiltonian system, represent a row-vector formed by the partial derivatives of the Hamiltonian scalar function H with respect to each one of the components of the derivation variable. Equation (3) is nothing more than the control system (2); equation (4) is known as the* adjoint system*.*

Definition 3 *A quadruple* $(\mathbf{x}(\cdot),\mathbf{u}(\cdot),\psi_0,\boldsymbol{\psi}(\cdot))$ *satisfying the Pontryagin maximum principle is said to be a (Pontryagin)* extremal*. An extremal is* normal *when* $\psi_0 \neq 0$*,* abnormal *if* $\psi_0 = 0$*.*

Remark 4 *Since we assume* Ω *to be an open set, the maximality condition* (5) *implies the* stationary condition

$$\frac{\partial H}{\partial \mathbf{u}}(t,\mathbf{x}(t),\mathbf{u}(t),\psi_0,\boldsymbol{\psi}(t)) = \mathbf{0}\,, \quad t \in [a,b]\,. \tag{7}$$

Using the Hamiltonian system (3)-(4)*, and the stationary condition* (7)*, it follows that, along the extremals, the total derivative of the Hamiltonian with respect to* t *equals its partial derivative,* $t \in [a,b]$*:*

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t,\mathbf{x}(t),\mathbf{u}(t),\psi_0,\boldsymbol{\psi}(t)) = \frac{\partial H}{\partial t}(t,\mathbf{x}(t),\mathbf{u}(t),\psi_0,\boldsymbol{\psi}(t))\,. \tag{8}$$

Let us now consider a one-parameter group of $\mathbb{C}^1$ transformations $\mathbf{h}^s : [a,b]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n$ of the form

$$\mathbf{h}^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) = (h_t^s(t,\mathbf{x},\psi_0,\boldsymbol{\psi}),\mathbf{h}_{\mathbf{x}}^s(t,\mathbf{x},\psi_0,\boldsymbol{\psi}),\mathbf{h}_{\mathbf{u}}^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}),\mathbf{h}_{\boldsymbol{\psi}}^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}))\,, \tag{9}$$

which reduces to the identity transformation when the parameter s vanishes:

$$h_t^0(t,\mathbf{x},\psi_0,\boldsymbol{\psi}) = t,\ \mathbf{h}_{\mathbf{x}}^0(t,\mathbf{x},\psi_0,\boldsymbol{\psi}) = \mathbf{x},\ \mathbf{h}_{\mathbf{u}}^0(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) = \mathbf{u},\ \mathbf{h}_{\boldsymbol{\psi}}^0(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) = \boldsymbol{\psi}\,.$$

Associated with a one-parameter group of transformations (9), we introduce the infinitesimal *generators*

$$T(t,\mathbf{x},\psi_0,\boldsymbol{\psi}) = \left.\frac{\partial}{\partial s} h_t^s(t,\mathbf{x},\psi_0\,\boldsymbol{\psi})\right|_{s=0}, \ \mathbf{X}(t,\mathbf{x},\psi_0,\boldsymbol{\psi}) = \left.\frac{\partial}{\partial s}\mathbf{h}_{\mathbf{x}}^s(t,\mathbf{x},\psi_0,\boldsymbol{\psi})\right|_{s=0},$$
$$\mathbf{U}(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) = \left.\frac{\partial}{\partial s}\mathbf{h}_{\mathbf{u}}^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})\right|_{s=0}, \ \boldsymbol{\Psi}(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) = \left.\frac{\partial}{\partial s}\mathbf{h}_{\boldsymbol{\psi}}^s(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi})\right|_{s=0} \tag{10}$$

Emmy Noether was the first who established a relation between the existence of invariance transformations of the problem and the existence of conservation laws [14]. This relation constitutes a universal principle that can be formulated, as a theorem, in several different contexts, under several different hypotheses (see e.g. [4, 6, 9, 10, 15, 26, 28]). Contributions in the literature go, however, further than extending Noether's theorem to different contexts, and weakening its assumptions. Since the pioneering work by Noether [14], several definitions of invariance have been introduced for the problems of the calculus of variations (see e.g. [10, 11, 20, 26]); and for the problems of optimal control (see e.g. [4, 6, 23, 27]). All these definitions are given with respect to a one-parameter group of transformations (9). Although written in a different way (some of these invariance/symmetry notions involve the integral functional, others only the integrand; some of them involve the original problem and the transformed one, others only the rate of change with respect to the parameter; etc) it turns out that, when written in terms of the generators (10), one gets a necessary and sufficient condition of invariance that, essentially, coincides with all those definitions. For this reason, here we define invariance directly in terms of the generators (10).

Definition 5 (([4, 28])) *We say that an optimal control problem (1)-(2) is* invariant *under* (10) *or, equivalently, that* (10) *is a* symmetry *of the problem, if, and only if,*

$$\frac{\partial H}{\partial t}T + \frac{\partial H}{\partial \mathbf{x}} \cdot \mathbf{X} + \frac{\partial H}{\partial \mathbf{u}} \cdot \mathbf{U} + \frac{\partial H}{\partial \boldsymbol{\psi}} \cdot \boldsymbol{\Psi} - \boldsymbol{\Psi}^T \cdot \dot{\mathbf{x}} - \boldsymbol{\psi}^T \cdot \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} + H\frac{\mathrm{d}T}{\mathrm{d}t} = 0\,, \tag{11}$$

with H the Hamiltonian (6)*.*

A symmetry is an intrinsic property of the optimal control problem (1)-(2) (an intrinsic property of the corresponding Hamiltonian (6)), and does not depend on the extremals. If one restricts attention to the quadruples $(\mathbf{x}(\cdot), \mathbf{u}(\cdot), \psi_0, \boldsymbol{\psi}(\cdot))$ that satisfy the Hamiltonian system and the maximality condition, one arrives at E. Noether's theorem: along the extremals, equalities (3), (4), (7), and (8) permit to simplify (11) to the form

$$\frac{\mathrm{d}H}{\mathrm{d}t}T - \dot{\boldsymbol{\psi}}^{\mathrm{T}} \cdot \mathbf{X} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} + H\frac{\mathrm{d}T}{\mathrm{d}t} = 0 \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(HT - \boldsymbol{\psi}^T \cdot \mathbf{X}\right) = 0\,. \tag{12}$$

Definition 6 *A function $C(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi})$ which is preserved along all the extremals of the optimal control problem and all $t \in [a, b]$,*

$$C(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t)) = const\,, \tag{13}$$

is called a first integral. *Equation* (13) *is said to be a* conservation law.

Equation (12) asserts that $C = HT - \boldsymbol{\psi}^T \cdot \mathbf{X}$ is a first integral:

Theorem 7 ((Noether's theorem)) *If (10) is a* symmetry *of problem (1)-(2), then*

$$\boldsymbol{\psi}(t)^{\mathrm{T}} \cdot \mathbf{X}(t, \mathbf{x}(t), \psi_0, \boldsymbol{\psi}(t)) - H(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t))T(t, \mathbf{x}(t), \psi_0, \boldsymbol{\psi}(t)) = const \tag{14}$$

is a conservation law.

From expression (14) we see that Noetherian conservation laws, associated with a certain optimal control problem, that is, with a certain Hamiltonian $H(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi})$, only depend on the generators T and $\mathbf{X}$ of a symmetry $(T, \mathbf{X}, \mathbf{U}, \boldsymbol{\Psi})$ (10).

3 Computation of Conservation Laws

The conservation laws we are looking for are obtained by substitution of the components T and $\mathbf{X}$ of a symmetry of the problem in (14). In section 6 we introduce the Maple procedure *Noether* to do these calculations for us. The input to this procedure is: the Lagrangian L and the velocity vector $\boldsymbol{\varphi}$, that define the optimal control problem (1)-(2) and the respective Hamiltonian H; and a symmetry, or a family of symmetries, obtained by means of our procedure *Symmetry* (see the procedure *Symmetry* in §6). The output of *Noether* is the corresponding conservation law (14). The non-trivial part of the computation lies in the determination of the symmetries of the problem (implemented in the Maple procedure *Symmetry*). Our algorithm for determining the infinitesimal generators is based on the necessary and sufficient condition of invariance (11). The key to do the calculations consists in observing that when we substitute the Hamiltonian H and its partial derivatives in the invariance identity (11), then the condition becomes a polynomial in $\dot{\mathbf{x}}$ and $\dot{\boldsymbol{\psi}}$, and one can set the coefficients equal to zero. Let us see how it works in detail.

Substituting H and its partial derivatives into (11), and expanding the total derivatives

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial T}{\partial \boldsymbol{\psi}} \cdot \dot{\boldsymbol{\psi}}, \quad \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \frac{\partial \mathbf{X}}{\partial t} + \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial \mathbf{X}}{\partial \boldsymbol{\psi}} \cdot \dot{\boldsymbol{\psi}},$$

one can write equation (11) as a polynomial

$$A(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) + B(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) \cdot \dot{\mathbf{x}} + C(t,\mathbf{x},\mathbf{u},\psi_0,\boldsymbol{\psi}) \cdot \dot{\boldsymbol{\psi}} = 0 \tag{15}$$

in the $2n$ derivatives $\dot{\mathbf{x}}$ and $\dot{\boldsymbol{\psi}}$:

$$\begin{aligned} &\left(\frac{\partial H}{\partial t} T + \frac{\partial H}{\partial \mathbf{x}} \cdot \mathbf{X} + \frac{\partial H}{\partial \mathbf{u}} \cdot \mathbf{U} + \frac{\partial H}{\partial \boldsymbol{\psi}} \cdot \boldsymbol{\Psi} + H\frac{\partial T}{\partial t} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\partial \mathbf{X}}{\partial t}\right) \\ &\quad + \left(-\boldsymbol{\Psi}^{\mathrm{T}} + H\frac{\partial T}{\partial \mathbf{x}} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right) \cdot \dot{\mathbf{x}} + \left(H\frac{\partial T}{\partial \boldsymbol{\psi}} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\partial \mathbf{X}}{\partial \boldsymbol{\psi}}\right) \cdot \dot{\boldsymbol{\psi}} = 0\,. \end{aligned} \tag{16}$$

The terms in (16) which involve derivatives with respect to vectors are expanded in row-vectors or in matrices, depending, respectively, if the function is a scalar or a vectorial one. For example,

$$\begin{aligned} \frac{\partial T}{\partial \mathbf{x}} &= \begin{bmatrix} \frac{\partial T}{\partial x_1} & \frac{\partial T}{\partial x_2} & \cdots & \frac{\partial T}{\partial x_n} \end{bmatrix}, \\ \frac{\partial \mathbf{X}}{\partial \boldsymbol{\psi}} &= \begin{bmatrix} \frac{\partial \mathbf{X}}{\partial \psi_1} & \frac{\partial \mathbf{X}}{\partial \psi_2} & \cdots & \frac{\partial \mathbf{X}}{\partial \psi_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial X_1}{\partial \psi_1} & \frac{\partial X_1}{\partial \psi_2} & \cdots & \frac{\partial X_1}{\partial \psi_n} \\ \frac{\partial X_2}{\partial \psi_1} & \frac{\partial X_2}{\partial \psi_2} & \cdots & \frac{\partial X_2}{\partial \psi_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial X_n}{\partial \psi_1} & \frac{\partial X_n}{\partial \psi_2} & \cdots & \frac{\partial X_n}{\partial \psi_n} \end{bmatrix}. \end{aligned}$$

Given an optimal control problem, defined by a Lagrangian L and a control system (2), we determine the infinitesimal generators T, $\mathbf{X}$, $\mathbf{U}$ and $\boldsymbol{\Psi}$, which define a symmetry for the problem, by the following method. Equation (16) is a differential equation in the $2n+m+1$ unknown functions T, X_1, …, X_n, U_1, …, U_m, Ψ_1, …, and Ψ_n. This equation must hold for all $\dot{x}_1$, …, $\dot{x}_n$, $\dot{\psi}_1$, …, $\dot{\psi}_n$, and therefore the coefficients A, B, and C of polynomial (15) must vanish, that is,

$$\begin{cases} \frac{\partial H}{\partial t}T + \frac{\partial H}{\partial \mathbf{x}} \cdot \mathbf{X} + \frac{\partial H}{\partial \mathbf{u}} \cdot \mathbf{U} + \frac{\partial H}{\partial \boldsymbol{\psi}} \cdot \boldsymbol{\Psi} + H\frac{\partial T}{\partial t} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\partial \mathbf{X}}{\partial t} = 0\,, \\ -\boldsymbol{\Psi}^{\mathrm{T}} + H\frac{\partial T}{\partial \mathbf{x}} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \mathbf{0}\,, \\ H\frac{\partial T}{\partial \boldsymbol{\psi}} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\partial \mathbf{X}}{\partial \boldsymbol{\psi}} = \mathbf{0}\,. \end{cases} \tag{17}$$

System of equations (17), obtained from (16), is a system of $2n+1$ partial differential equations with $2n+m+1$ unknown functions (so, in general, there exists not a unique symmetry but a whole family of symmetries – see examples in §4 and §5). Although a system of PDEs, its solution is possible because the system is linear with respect to the unknown functions and their derivatives. However, when dealing with optimal control problems with several state and control variables, the number of calculations is big enough, and the help of the computer is more than welcome. Our Maple procedure *Symmetry*, in §6, does the job for us. The procedure receives, as input, the Lagrangian and the expressions which define the control system; and gives, as output, a family of symmetries $(T, \mathbf{X}, \mathbf{U}, \boldsymbol{\Psi})$. Since system (17) is homogeneous, we always have, as trivial solution, $(T, \mathbf{X}, \mathbf{U}, \boldsymbol{\Psi}) = \mathbf{0}$. This does not give any additional information (for the trivial solution, Noether's theorem is the truism "zero is a constant"). When the output of *Symmetry* coincides with the trivial solution, that means, roughly speaking, that the optimal control problem does not admit a symmetry (more precisely – see §6 – it means that our procedure Maple was not able to find symmetries for the problem).

Summarizing: given an optimal control problem (1)-(2) we compute conservation laws, in an automatic way, through two steps: (i) with our procedure *Symmetry* we obtain all the possible invariance symmetries of the problem; (ii) using the obtained symmetries as input to procedure *Noether*, based on Theorem 7, we obtain the correspondent conservation laws.

In the next two sections we give several examples illustrating the whole process.

4 Illustrative Examples

In order to show the functionality and the use of the routines developed, we apply our Maple package to several concrete optimal control problems found in the literature. The obtained results show the correctness and usefulness of the Maple code. All the computational processing was carried out with Maple version 9.5 on a typical 1.66 GHz PC of 512Mb RAM. The total computing time is indicated for each example in the format min'sec".

Example 8 *(0'02") Let us begin with the minimization of the functional* $\int_a^b L(u(t))\mathrm{d}t$ *subject to the control system* $\dot{x}(t) = \varphi(u(t))x(t)$. *This is a very simple problem, with one state variable* ($n = 1$) *and one control variable* ($m = 1$). *With Maple definitions*

```
> l:=L(u); Phi:=phi(u)*x;
```

$$\begin{aligned} l &:= L(u) \\ \Phi &:= \varphi(u)\,x \end{aligned}$$

our procedure Symmetry *determines the infinitesimal invariance generators of the optimal control problem under consideration:*

```
> Symmetry(l,Phi,t,x,u);
```

$$\{T = C_2,\ X = C_1 x,\ U = 0,\ \Psi = -C_1\,\psi\}$$

The family of Conservation Laws associated with the generators just obtained, is easily obtained through our procedure Noether *(the sign of percentage – % – is an operator used in Maple to represent the result of the previous command):*

```
> Noether(l,Phi,t,x,u, %);
```

$$C_1 x(t)\psi(t) - (\psi_0 L(u(t)) + \psi(t)\varphi(u(t))\,x(t))\,C_2 = const$$

The obtained conservation law depends on two parameters. Since the problem is autonomous, the fact that the Hamiltonian $H = \psi_0 L(u(t)) + \psi(t)\varphi(u(t))\,x(t)$ *is constant along the extremals is a trivial consequence of the property* (8). *With the substitutions*

```
> subs(C[1]=1,C[2]=0, %);
```

$$x(t)\psi(t) = const$$

one gets the conservation law obtained in [24, Example 4].

Example 9 *(1'13") Let us consider now the following problem:*

$$\int_a^b \left(u_1(t)^2 + u_2(t)^2\right) \mathrm{d}t \longrightarrow \min,$$

$$\begin{cases} \dot{x}_1(t) = u_1(t)\cos x_3(t), \\ \dot{x}_2(t) = u_1(t)\sin x_3(t), \\ \dot{x}_3(t) = u_2(t), \end{cases}$$

where the control system serves as model for the kinematics of a car [12, Example 18, p. 750]. In this case the optimal control problem has three state variables ($n = 3$*) and two controls (*$m = 2$*). The conservation law for this example, and the next ones, is obtained by the same process followed in Example 8.*

```
> L:=u[1]^2+u[2]^2; phi:=[u[1]*cos(x[3]),u[1]*sin(x[3]),u[2]];
```

$$\begin{aligned} L &:= {u_1}^2 + {u_2}^2 \\ \varphi &:= [u_1\cos(x_3),\, u_1\sin(x_3),\, u_2] \end{aligned}$$

```
> Symmetry(L, phi, t, [x[1],x[2],x[3]], [u[1],u[2]]);
```

$$\{T = C_2,\, X_1 = -C_1 x_2 + C_3,\, X_2 = C_1 x_1 + C_4,\, X_3 = C_1,\, U_1 = 0,\, U_2 = 0,\, \Psi_1 = -C_1 \psi_2,\, \Psi_2 = C_1 \psi_1,\, \Psi_3 = 0\}$$

```
> Noether(L, phi, t, [x[1],x[2],x[3]], [u[1],u[2]], %);
```

$$(-C_1 x_2(t) + C_3)\, \psi_1(t) + (C_1 x_1(t) + C_4)\, \psi_2(t) + C_1 \psi_3(t) - \left(\psi_0 \left((u_1(t))^2 + (u_2(t))^2\right) + u_1(t) \cos(x_3(t))\, \psi_1(t) + u_1(t) \sin(x_3(t))\, \psi_2(t) + u_2(t) \psi_3(t)\right) C_2 = const$$

Choosing $C_1 = 1$ and $C_2 = C_3 = C_4 = 0$ we obtain, from Theorem 7, the conservation law

```
> subs(C[1]=1,C[2]=0,C[3]=0,C[4]=0, %);
```

$$-x_2(t)\psi_1(t) + x_1(t)\psi_2(t) + \psi_3(t) = const$$

which corresponds to the symmetry group of planar (orientation-preserving) isometries given in [12, Example 18, p. 750].

Example 10 *(0'01") Let us return to a scalar problem ($n = m = 1$):*

$$\int_a^b e^{tx(t)} u(t) dt \longrightarrow \min, \quad \dot{x}(t) = t x(t) u(t)^2 \,.$$

```
> L:=exp(t*x)*u; phi:=t*x*u^2;
```

$$L := e^{tx} u$$
$$\varphi := t x u^2$$

```
> Symmetry(L, phi, t, x, u);
```

$$\{T = -tC_1,\, X = C_1 x,\, U = C_1 u,\, \Psi = -\psi\, C_1\}$$

```
> Noether(L, phi, t, x, u, %);
```

$$C_1 x(t) \psi(t) + \left(\psi_0 e^{tx(t)} u(t) + \psi(t) t x(t) (u(t))^2\right) t C_1 = const$$

By choosing $C_1 = 1$

```
> expand(subs(C[1]=1, %));
```

$$x(t)\psi(t) + t\psi_0 e^{tx(t)} u(t) + \psi(t) t^2 x(t) (u(t))^2 = const$$

one obtains the conservation law of [25, Example 1].

Example 11 *(6'41") We now consider an optimal control problem with four state variables ($n = 4$) and two controls ($m = 2$):*

$$\int_a^b \left(u_1(t)^2 + u_2(t)^2\right) \mathrm{d}t \longrightarrow \min,$$

$$\begin{cases} \dot{x}_1(t) = x_3(t), \\ \dot{x}_2(t) = x_4(t), \\ \dot{x}_3(t) = -x_1(t)\left(x_1(t)^2 + x_2(t)^2\right) + u_1(t), \\ \dot{x}_4(t) = -x_2(t)\left(x_1(t)^2 + x_2(t)^2\right) + u_2(t), \end{cases}$$

```
> L:=u[1]^2+u[2]^2; phi:=[x[3],x[4],-x[1]*(x[1]^2+x[2]^2)+u[1],
    -x[2]*(x[1]^2+x[2]^2)+u[2]];
```

$$\begin{aligned} L &:= {u_1}^2 + {u_2}^2 \\ \varphi &:= \left[x_3, x_4, -x_1\left({x_1}^2 + {x_2}^2\right) + u_1, -x_2\left({x_1}^2 + {x_2}^2\right) + u_2\right] \end{aligned}$$

```
> Symmetry(L, phi, t, [x[1],x[2],x[3],x[4]], [u[1],u[2]]);
```

$$\Big\{T = C_3,\, X_1 = C_1 x_2,\, X_2 = -C_1 x_1,\, X_3 = C_1 x_4, X_4 = -C_1 x_3,$$
$$U_1 = -C_2 u_2 - \frac{1}{2}\frac{(C_2 + C_1)\,\psi_4}{\psi_0},\, U_2 = C_2 u_1 + \frac{1}{2}\frac{(C_2 + C_1)\,\psi_3}{\psi_0},$$
$$\Psi_1 = C_1\psi_2,\, \Psi_2 = -C_1\psi_1,\, \Psi_3 = C_1\psi_4,\, \Psi_4 = -C_1\psi_3\Big\}$$

```
> Noether(L, phi, t, [x[1],x[2],x[3],x[4]], [u[1],u[2]], %);
```

$$\begin{aligned} C_1 x_2(t)\psi_1(t) - C_1 x_1(t)\psi_2(t) + C_1 x_4(t)\psi_3(t) - C_1 x_3(t)\psi_4(t) - \Big(\psi_0\left((u_1(t))^2 + (u_2(t))^2\right) \\ + x_3(t)\psi_1(t) + x_4(t)\psi_2(t) + \Big(-x_1(t)\left((x_1(t))^2 + (x_2(t))^2\right) + u_1(t)\Big)\,\psi_3(t) \\ + \Big(-x_2(t)\left((x_1(t))^2 + (x_2(t))^2\right) + u_2(t)\Big)\,\psi_4(t)\Big) C_3 = \mathit{const} \end{aligned}$$

The substitutions

```
> subs(C[1]=-1,C[3]=0, %);
```

$$-x_2(t)\psi_1(t) + x_1(t)\,\psi_2(t) - x_4(t)\,\psi_3(t) + x_3(t)\psi_4(t) = \mathit{const}$$

conduce us to the conservation law provided in [27, Example 5.2].

Example 12 *(6'42") Another problem with $n = 4$ and $m = 2$:*

$$\int_a^b \left(u_1(t)^2 + u_2(t)^2\right) \mathrm{d}t \longrightarrow \min,$$

$$\begin{cases} \dot{x}_1(t) = u_1(t)\left(1 + x_2(t)\right), \\ \dot{x}_2(t) = u_1(t)x_3(t), \\ \dot{x}_3(t) = u_2(t), \\ \dot{x}_4(t) = u_1(t)x_3(t)^2. \end{cases}$$

```
> L:=u[1]^2+u[2]^2; phi:=[u[1]*(1+x[2]),u[1]*x[3],u[2],u[1]*x[3]^2];
```

$$L := {u_1}^2 + {u_2}^2$$
$$\varphi := [u_1\left(1 + x_2\right), u_1 x_3, u_2, u_1 {x_3}^2]$$

```
> Symmetry(L, phi, t, [x[1],x[2],x[3],x[4]], [u[1],u[2]]);
```

$$\Big\{T = \frac{2}{3}\, C_1 t + C_2,\ X_1 = C_1 x_1 + C_4,\ X_2 = \frac{2}{3}\, C_1 + \frac{2}{3}\, C_1 x_2,\ X_3 = \frac{1}{3}\, C_1 x_3,\ X_4 = C_1 x_4 + C_3,$$
$$U_1 = -\frac{1}{3}\, u_1 C_1,\ U_2 = -\frac{1}{3}\, C_1 u_2,\ \Psi_1 = -C_1 \psi_1,\ \Psi_2 = -\frac{2}{3}\, C_1 \psi_2,\ \Psi_3 = -\frac{1}{3}\, C_1 \psi_3,\ \Psi_4 = -C_1 \psi_4\Big\}$$

```
> Noether(L, phi, t, [x[1],x[2],x[3],x[4]], [u[1],u[2]], %);
```

$$\left(C_1 x_1(t) + C_4\right)\psi_1(t) + \left(\frac{2}{3}\, C_1 + \frac{2}{3}\, C_1 x_2(t)\right)\psi_2(t) + \frac{1}{3}\, C_1 x_3(t)\psi_3(t) + \left(C_1 x_4(t) + C_3\right)\psi_4(t)$$
$$- \Bigg(\psi_0\left((u_1(t))^2 + (u_2(t))^2\right) + \psi_1(t)u_1(t)\left(1 + x_2(t)\right) + \psi_2(t)u_1(t)x_3(t)$$
$$+ \psi_3(t)u_2(t) + \psi_4(t)u_1(t)\left(x_3(t)\right)^2\Bigg)\left(\frac{2}{3}\, C_1 t + C_2\right) = const$$

With the substitutions

```
> subs(C[1]=3,C[2]=0,C[3]=0,C[4]=0, %);
```

$$3\, x_1(t)\psi_1(t) + \left(2 + 2\, x_2(t)\right)\psi_2(t) + x_3(t)\psi_3(t) + 3\, x_4(t)\psi_4(t) - 2\,\Big(\psi_0\left((u_1(t))^2 + (u_2(t))^2\right)$$
$$+ \psi_1(t)u_1(t)\left(1 + x_2(t)\right) + \psi_2(t)u_1(t)x_3(t) + \psi_3(t)u_2(t) + \psi_4(t)u_1(t)\left(x_3(t)\right)^2\Big)t = const$$

we have the conservation law obtained in [27, Example 5.3].

Example 13 *(0'04") Let us consider*

$$\int_a^b u(t)^2 \mathrm{d}t \longrightarrow \min,$$

$$\begin{cases} \dot{x}(t) = 1 + y(t)^2, \\ \dot{y}(t) = u(t). \end{cases}$$

```
> L:=u^2; phi:=[1+y^2,u];
```

$$L := u^2$$
$$\varphi := [1+y^2, u]$$

```
> Symmetry(L, phi, t, [x,y], u);
```

$$\left\{T = \frac{1}{2}C_1 t + C_2,\ X_1 = -\frac{1}{2}C_1 t + C_1 x + C_3,\ X_2 = \frac{1}{4}C_1 y,\ U = -\frac{1}{4}C_1 u,\ \Psi_1 = -C_1\psi_1,\ \Psi_2 = -\frac{1}{4}C_1\psi_2\right\}$$

```
> Noether(L, phi, t, [x,y], u, %);
```

$$\left(-\frac{1}{2}C_1 t + C_1 x(t) + C_3\right)\psi_1(t) + \frac{1}{4}C_1 y(t)\psi_2(t) - \left(\psi_0 (u(t))^2 + \psi_1(t)\left(1 + (y(t))^2\right) + \psi_2(t)u(t)\right)\left(\frac{1}{2}C_1 t + C_2\right) = const$$

From substitutions

```
> subs(C[1]=-4,C[2]=0,C[3]=0,%);
```

$$(2\,t - 4\,x(t))\,\psi_1(t) - y(t)\psi_2(t) + 2\left(\psi_0 (u(t))^2 + \psi_1(t)\left(1 + (y(t))^2\right) + \psi_2(t)u(t)\right) t = const$$

we obtain the conservation law in [27, Example 6.2].

Example 14 *(2'44") We consider now a minimum time problem ($T \to \min \Leftrightarrow \int_0^T 1\mathrm{d}t \to \min$) with the following control system:*

$$\begin{cases} \dot{x}_1(t) = 1 + x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = u(t), \\ \dot{x}_4(t) = x_3(t)^2 - x_2(t)^2. \end{cases}$$

In Maple we have:

```
> L:=1; phi:=[1+x[2],x[3],u,x[3]^2-x[2]^2];
```

$$L := 1$$
$$\varphi := [1 + x_2, x_3, u, {x_3}^2 - {x_2}^2]$$

```
> Symmetry(L, phi, t, [x[1],x[2],x[3],x[4]], u);
```

$$\Big\{T = C_5,\ X_1 = \left(-\frac{1}{2}\,C_2 - \frac{1}{2}\,C_1\right) t + \frac{1}{2}\,C_2 x_1 + C_4,\ X_2 = -\frac{1}{2}\,C_1 + \frac{1}{2}\,C_2 x_2,\ X_3 = \frac{1}{2}\,C_2 x_3,\ X_4 = -C_1 t + C_1 x_1 + C_2 x_4 + C_3,\ U = \frac{1}{2}\,u C_2,\ \Psi_1 = -\frac{1}{2}\,C_2\psi_1 - C_1\psi_4,\ \Psi_2 = -\frac{1}{2}\,C_2\psi_2,\ \Psi_3 = -\frac{1}{2}\,C_2\psi_3,\ \Psi_4 = -C_2\psi_4\Big\}$$

```
> Noether(L, phi, t, [x[1],x[2],x[3],x[4]], u, %);
```

$$\left(\left(-\frac{1}{2}\,C_2 - \frac{1}{2}\,C_1\right) t + \frac{1}{2}\,C_2 x_1(t) + C_4\right)\psi_1(t) + \left(-\frac{1}{2}\,C_1 + \frac{1}{2}\,C_2 x_2(t)\right)\psi_2(t) + \frac{1}{2}\,C_2 x_3(t)\psi_3(t) + \left(-C_1 t + C_1 x_1(t) + C_2 x_4(t) + C_3\right)\psi_4(t) - \left(\psi_0 + (1 + x_2(t))\,\psi_1(t) + x_3(t)\psi_2(t) + \psi_3(t)u(t) + \left((x_3(t))^2 - (x_2(t))^2\right)\psi_4(t)\right) C_5 = const$$

With the appropriate values for the constants,

```
> subs(C[1]=0,C[2]=2,C[3]=0,C[4]=0,C[5]=0,%);
```

$$(-t + x_1(t))\,\psi_1(t) + x_2(t)\psi_2(t) + x_3(t)\psi_3(t) + 2\,x_4(t)\psi_4(t) = const$$

we obtain the conservation law derived in [27, Example 6.3].

Example 15 *(0'25") Follows another problem of minimum time, with control system given by*

$$\begin{cases} \dot{x}(t) = 1 + y(t)^2 - z(t)^2\,, \\ \dot{y}(t) = z(t)\,, \\ \dot{z}(t) = u(t)\,. \end{cases}$$

```
> L:=1; phi:=[1+y^2-z^2,z,u];
```

$$L := 1$$
$$\varphi := [1 + y^2 - z^2, z, u]$$

```
> Symmetry(L, phi, t, [x,y,z], u);
```

$$\left\{T = C_2,\ X_1 = -C_1 t + C_1 x + C_3,\ X_2 = \frac{1}{2}\,C_1 y,\ X_3 = \frac{1}{2}\,C_1 z,\right.$$
$$\left. U = \frac{1}{2}\,C_1 u,\ \Psi_1 = -C_1\psi_1,\ \Psi_2 = -\frac{1}{2}\,C_1\psi_2,\ \Psi_3 = -\frac{1}{2}\,C_1\psi_3\right\}$$

```
> Noether(L, phi, t, [x,y,z], u, %);
```

$$(-C_1 t + C_1 x(t) + C_3)\,\psi_1(t) + \frac{1}{2}\,C_1 y(t)\psi_2(t) + \frac{1}{2}\,C_1 z(t)\psi_3(t)$$
$$-\left(\psi_0 + \psi_1(t)\left(1 + (y(t))^2 - (z(t))^2\right) + \psi_2(t)z(t) + \psi_3(t)u(t)\right)C_2 = const$$

Substitutions

```
> subs(C[1]=2,C[2]=0,C[3]=0, %);
```

$$(-2\,t + 2\,x(t))\,\psi_1(t) + y(t)\psi_2(t) + z(t)\psi_3(t) = const$$

specify the conservation law in the one obtained in [27, Example 6.4].

We finish the section by applying our Maple package to three important problems of geodesics in sub-Riemannian geometry. The reader, interested in the study of symmetries of flat distributions of sub-Riemannian geometry, is referred to [19].

Example 16 (Martinet – $(\mathbf{2},\mathbf{2},\mathbf{3})$ problem) *Given the problem ($n = 3, m = 2$)*

$$\int_a^b \left(u_1(t)^2 + u_2(t)^2\right) \mathrm{d}t \longrightarrow \min,$$
$$\begin{cases} \dot{x}_1(t) = u_1(t), \\ \dot{x}_2(t) = \dfrac{u_2(t)}{1 + \alpha x_1(t)}, & \alpha \in \mathbb{R}, \\ \dot{x}_3(t) = x_2(t)^2 u_1(t), \end{cases}$$

we consider two distinct situations: $\alpha = 0$ (Martinet problem of sub-Riemannian geometry in the flat case – see [2]) and $\alpha \neq 0$ (non-flat case).

Flat Problem ($\alpha = 0$, 1'06"):

```
> L:=u[1]^2+u[2]^2; phi:=[u[1],u[2],x[2]^2*u[1]];
```

$$L := {u_1}^2 + {u_2}^2$$
$$\varphi := [u_1, u_2, {x_2}^2 u_1]$$

```
> Symmetry(L, phi, t, [x[1],x[2],x[3]], [u[1],u[2]]);
```

$$\left\{T = \frac{2}{3}\,C_1 t + C_2,\ X_1 = \frac{1}{3}\,C_1 x_1 + C_4,\ X_2 = \frac{1}{3}\,C_1 x_2,\ X_3 = C_1 x_3 + C_3,\right.$$
$$\left.U_1 = -\frac{1}{3}\,u_1 C_1,\ U_2 = -\frac{1}{3}\,C_1 u_2,\ \Psi_1 = -\frac{1}{3}\,C_1 \psi_1,\ \Psi_2 = -\frac{1}{3}\,C_1 \psi_2,\ \Psi_3 = -C_1 \psi_3\right\}$$

```
> Noether(L, phi, t, [x[1],x[2],x[3]], [u[1],u[2]], %);
```

$$\left(\frac{1}{3}\,C_1 x_1(t) + C_4\right)\psi_1(t) + \frac{1}{3}\,C_1 x_2(t)\psi_2(t) + \left(C_1 x_3(t) + C_3\right)\psi_3(t)$$
$$-\left(\psi_0\left((u_1(t))^2 + (u_2(t))^2\right) + \psi_1(t)u_1(t) + \psi_2(t)u_2(t) + \psi_3(t)\,(x_2(t))^2\,u_1(t)\right)\left(\frac{2}{3}\,C_1 t + C_2\right) = const$$

With the substitutions

```
> subs(C[1]=3,C[2]=0,C[3]=0,C[4]=0, %);
```

$$x_1(t)\psi_1(t) + x_2(t)\psi_2(t) + 3\,x_3(t)\psi_3(t)$$
$$-\,2\left(\psi_0\left((u_1(t))^2 + (u_2(t))^2\right) + \psi_1(t)u_1(t) + \psi_2(t)u_2(t) + \psi_3(t)\,(x_2(t))^2\,u_1(t)\right)t = const$$

we get the conservation law first obtained in [25, Example 2].

Non-Flat Problem ($\alpha \neq 0$, 1'14"):

```
> L:=u[1]^2+u[2]^2; phi:=[u[1],u[2]/(1+alpha*x[1]),x[2]^2*u[1]];
```

$$L := {u_1}^2 + {u_2}^2$$
$$\varphi := \left[u_1, \frac{u_2}{1+\alpha\,x_1}, {x_2}^2 u_1\right]$$

```
> simplify(Symmetry(L, phi, t, [x[1],x[2],x[3]], [u[1],u[2]]));
```

$$\left\{T = 2\,C_7 t + C_{11},\ X_1 = C_7(\alpha^{-1} + x_1),\ X_2 = 0,\ X_3 = C_7 x_3 + C_{10},\right.$$
$$\left.U_1 = -C_7 u_1,\ U_2 = -C_7 u_2,\ \Psi_1 = -C_7 \psi_1,\ \Psi_2 = 0,\ \Psi_3 = -C_7 \psi_3\right\}$$

```
> Noether(L, phi, t, [x[1],x[2],x[3]], [u[1],u[2]], %);
```

$$C_7(\alpha^{-1} + x_1(t))\psi_1(t) + \left(C_7 x_3(t) + C_{10}\right)\psi_3(t)$$
$$-\left(\psi_0\left((u_1(t))^2 + (u_2(t))^2\right) + u_1(t)\psi_1(t) + \frac{u_2(t)\psi_2(t)}{1+\alpha\,x_1(t)} + (x_2(t))^2\,u_1(t)\psi_3(t)\right)\left(2\,C_7 t + C_{11}\right) = const$$

When $C_7 = 1$ and $C_{10} = C_{11} = 0$,

```
> subs(C[7]=1,C[10]=0,C[11]=0, %);
```

$$\left(\alpha^{-1}+x_1(t)\right)\psi_1(t)+x_3(t)\psi_3(t) \\ -2\left(\psi_0\left((u_1(t))^2+(u_2(t))^2\right)+u_1(t)\psi_1(t)+\frac{u_2(t)\psi_2(t)}{1+\alpha\,x_1(t)}+(x_2(t))^2\,u_1(t)\psi_3(t)\right)t=const$$

we obtain the conservation law proved in [23, Example 2].

Example 17 (Heisenberg – $(\mathbf{2},\mathbf{3})$ problem) *(1'04") The Heisenberg $(2,3)$ problem can be formulated as follows:*

$$\frac{1}{2}\int_a^b\left(u_1(t)^2+u_2(t)^2\right)\mathrm{d}t\longrightarrow\min,$$

$$\begin{cases}\dot{x}_1(t)=u_1(t)\,,\\ \dot{x}_2(t)=u_2(t)\,,\\ \dot{x}_3(t)=u_2(t)x_1(t)\,.\end{cases}$$

The problem was proved to be completely integrable using three independent conservation laws [22]. Such conservation laws can now be easily obtained with our Maple functions.

```
> L:=1/2*(u[1]^2+u[2]^2);
> phi:=[u[1], u[2], u[2]*x[1]];
```

$$L := \frac{1}{2}u_1{}^2+\frac{1}{2}u_2{}^2$$
$$\varphi := [u_1,u_2,u_2x_1]$$

```
> Symmetry(L, phi, t, [x[1],x[2],x[3]], [u[1],u[2]]);
```

$$\{T=2\,C_2t+C_5,\ X_1=C_1+C_2x_1,\ X_2=C_2x_2+C_3,\ X_3=C_1x_2+2\,C_2x_3+C_4,\\ U_1=-C_2u_1,\ U_2=-C_2u_2,\ \Psi_1=-C_2\psi_1,\ \Psi_2=-C_2\psi_2-\psi_3C_1,\ \Psi_3=-2\,C_2\psi_3\}$$

```
> CL:=Noether(L, phi, t, [x[1],x[2],x[3]], [u[1],u[2]], %);
```

$$CL := (C_1+C_2x_1(t))\,\psi_1(t)+(C_2x_2(t)+C_3)\,\psi_2(t)+(C_1x_2(t)+2\,C_2x_3(t)+C_4)\,\psi_3(t) \\ -\left(\psi_0\left(\frac{1}{2}(u_1(t))^2+\frac{1}{2}(u_2(t))^2\right)+\psi_1(t)u_1(t)+\psi_2(t)u_2(t)+\psi_3(t)u_2(t)x_1(t)\right)(2\,C_2t+C_5)=const$$

We now want to eliminate the controls from the previous family of conservation laws. We begin to define the Hamiltonian:

```
> H:=-L+Vector[row]([psi[1](t), psi[2](t), psi[3](t)]).Vector(phi);
```

$$H := -\frac{1}{2}u_1{}^2-\frac{1}{2}u_2{}^2+u_1\psi_1(t)+u_2\psi_2(t)+u_2x_1\psi_3(t)$$

The stationary condition (7) *permits to obtain the pair of controls* $(u_1(t),u_2(t))$*.*

```
> solve({diff(H,u[1])=0, diff(H,u[2])=0}, {u[1],u[2]}):
> subs(x[1]=x[1](t), u[1]=u[1](t), u[2]=u[2](t), %);
```

$$\{u_1(t) = \psi_1(t), u_2(t) = \psi_2(t) + x_1(t)\psi_3(t)\}$$

It is not difficult to show that the problem does not admit abnormal extremals, so one can choose, without any loss of generality, $\psi_0 = -1$.

```
> CL:=subs(psi[0]=-1, %, CL);
```

$$CL := (C_1 + C_2x_1(t))\,\psi_1(t) + (C_2x_2(t) + C_3)\,\psi_2(t) + (C_1x_2\,(t) + 2\,C_2x_3(t) + C_4)\,\psi_3(t) - \left(\frac{1}{2}\,(\psi_1(t))^2 - \frac{1}{2}\,(\psi_2(t) + x_1(t)\psi_3(t))^2 + \psi_2(t)\,(\psi_2(t) + x_1\,(t)\,\psi_3(t)) + \psi_3\,(t)\,(\psi_2(t) + x_1\,(t)\,\psi_3(t))\,x_1\,(t)\right)(2\,C_2t + C_5) = const$$

It is easy to extract from the family of conservation laws just obtained, three independent conservation laws. We just need to fix one constant to a non-zero value, and choose all the other constants to be zero:

```
> subs(C[4]=1,seq(C[i]=0,i=1..5), CL);
> subs(C[1]=1,seq(C[i]=0,i=1..5), CL);
> simplify(subs(C[5]=-1,seq(C[i]=0,i=1..5), CL));
```

$$\psi_3(t) = const$$
$$x_2(t)\psi_3(t) + \psi_1(t) = const$$
$$\frac{1}{2}\,\psi_1(t)^2 + \frac{1}{2}\,\psi_2(t)^2 + \psi_2(t)x_1(t)\psi_3(t) + \frac{1}{2}\,x_1(t)^2\psi_3(t)^2 = const$$

The last conservation law corresponds to the Hamiltonian. This, and $\psi_3 = const$, are trivial conservation laws for the problem. The missing first integral to solve the problem, $x_2\psi_3 + \psi_1$, was obtained in [22].

Example 18 (Cartan – $(\mathbf{2}, \mathbf{3}, \mathbf{5})$ problem) *(30'34") The Cartan problem with growth vector $(2, 3, 5)$ can be posed in the following way:*

$$\frac{1}{2}\int_a^b \left(u_1(t)^2 + u_2(t)^2\right) \mathrm{d}t \longrightarrow \min,$$

$$\begin{cases} \dot{x}_1(t) = u_1(t)\,, \\ \dot{x}_2(t) = u_2(t)\,, \\ \dot{x}_3(t) = u_2(t)x_1(t)\,, \\ \dot{x}_4(t) = \frac{1}{2}\,u_2(t)x_1(t)^2\,, \\ \dot{x}_5(t) = u_2(t)x_1(t)x_2(t)\,. \end{cases}$$

The integrability of the problem was recently established in [19]. This is possible with five independent conservation laws. They can easily be determined with our Maple package.

```
> L:=1/2*(u[1]^2+u[2]^2);
> phi:=[u[1], u[2], u[2]*x[1], 1/2*u[2]*x[1]^2, u[2]*x[1]*x[2]];
```

$$L := \frac{1}{2}{u_1}^2 + \frac{1}{2}{u_2}^2$$
$$\varphi := [u_1, u_2, u_2 x_1, \frac{1}{2} u_2 {x_1}^2, u_2 x_1 x_2]$$

```
> Symmetry(L, phi, t, [x[1],x[2],x[3],x[4],x[5]], [u[1],u[2]]);
```

$$\Big\{T = \frac{2}{3} C_1 t + C_4,\ X_1 = \frac{1}{3} C_1 x_1,\ X_2 = \frac{1}{3} C_1 x_2 + C_2,\ X_3 = \frac{2}{3} C_1 x_3 + C_6,$$
$$X_4 = C_1 x_4 + C_5,\ X_5 = C_2 x_3 + C_1 x_5 + C_3,\ U_1 = -\frac{1}{3} C_1 u_1,\ U_2 = -\frac{1}{3} C_1 u_2,$$
$$\Psi_1 = -\frac{1}{3} C_1 \psi_1,\ \Psi_2 = -\frac{1}{3} \psi_2 C_1,\ \Psi_3 = -\frac{2}{3} C_1 \psi_3 - C_2 \psi_5,\ \Psi_4 = -C_1 \psi_4,\ \Psi_5 = -C_1 \psi_5\Big\}$$

```
> CL:=Noether(L, phi, t, [x[1],x[2],x[3],x[4],x[5]], [u[1],u[2]], %);
```

$$CL := \frac{1}{3} C_1 x_1(t) \psi_1(t) + \left(\frac{1}{3} C_1 x_2(t) + C_2\right) \psi_2(t) + \left(\frac{2}{3} C_1 x_3(t) + C_6\right) \psi_3(t) + (C_1 x_4(t) + C_5)\, \psi_4(t)$$
$$+ (C_2 x_3(t) + C_1 x_5(t) + C_3)\, \psi_5(t) - \Bigg(\psi_0 \left(\frac{1}{2} (u_1(t))^2 + \frac{1}{2} (u_2(t))^2\right) + u_1(t)\psi_1(t) + u_2(t)\psi_2(t)$$
$$+ u_2(t) x_1(t) \psi_3(t) + \frac{1}{2} u_2(t) (x_1(t))^2 \psi_4(t) + u_2(t) x_1(t) x_2(t) \psi_5(t)\Bigg) \left(\frac{2}{3} C_1 t + C_4\right) = const$$

The Hamiltonian is given by

```
> H:=-L+Vector[row]([psi[1](t), psi[2](t), psi[3](t), psi[4](t),
                                              psi[5](t)]).Vector(phi);
```

$$H := -\frac{1}{2}{u_1}^2 - \frac{1}{2}{u_2}^2 + u_1 \psi_1(t) + u_2 \psi_2(t) + u_2 x_1 \psi_3(t) + \frac{1}{2} u_2 {x_1}^2 \psi_4(t) + u_2 x_1 x_2 \psi_5(t)$$

and the extremal controls are obtained through the stationary condition (7).

```
> solve({diff(H,u[1])=0, diff(H,u[2])=0}, {u[1],u[2]}):
> subs(x[1]=x[1](t),x[2]=x[2](t), u[1]=u[1](t),u[2]=u[2](t), %);
```

$$\left\{u_1(t) = \psi_1(t), u_2(t) = x_1(t) x_2(t) \psi_5(t) + \psi_2(t) + x_1(t) \psi_3(t) + \frac{1}{2} x_1(t)^2 \psi_4(t)\right\}$$

```
> CL:=subs(psi[0]=-1, %, CL);
```

$$
\begin{aligned}
CL := {} & \frac{1}{3}\,C_1 x_1(t)\psi_1(t) + \left(\frac{1}{3}\,C_1 x_2(t) + C_2\right)\psi_2(t) + \left(\frac{2}{3}\,C_1 x_3(t) + C_6\right)\psi_3(t) \\
& + \left(C_1 x_4(t) + C_5\right)\psi_4(t) + \left(C_2 x_3(t) + C_1 x_5(t) + C_3\right)\psi_5(t) - \Bigg(\frac{1}{2}\,\psi_1(t)^2 \\
& - \frac{1}{2}\left(x_1(t)x_2(t)\psi_5(t) + \psi_2(t) + x_1(t)\psi_3(t) + \frac{1}{2}\,x_1(t)^2\psi_4(t)\right)^2 \\
& + \left(x_1(t)x_2(t)\psi_5(t) + \psi_2(t) + x_1(t)\psi_3(t) + \frac{1}{2}\,x_1(t)^2\psi_4(t)\right)\psi_2(t) \\
& + \left(x_1(t)x_2(t)\psi_5(t) + \psi_2(t) + x_1(t)\psi_3(t) + \frac{1}{2}\,x_1(t)^2\psi_4(t)\right)x_1(t)\psi_3(t) \\
& + \frac{1}{2}\left(x_1(t)x_2(t)\psi_5(t) + \psi_2(t) + x_1(t)\psi_3(t) + \frac{1}{2}\,(x_1(t))^2\,\psi_4(t)\right)x_1(t)^2\psi_4(t) \\
& + \left(x_1(t)x_2(t)\psi_5(t) + \psi_2(t) + x_1(t)\psi_3(t) + \frac{1}{2}\,x_1(t)^2\psi_4(t)\right)x_1(t)x_2(t)\psi_5(t)\Bigg)\left(\frac{2}{3}\,C_1 t + C_4\right) = const
\end{aligned}
$$

The five conservation laws we are looking for, are easily obtained (the last one corresponds to the Hamiltonian):

```
> subs(C[6]=1,seq(C[i]=0,i=1..6), CL);
> subs(C[5]=1,seq(C[i]=0,i=1..6), CL);
> subs(C[3]=1,seq(C[i]=0,i=1..6), CL);
> subs(C[2]=1,seq(C[i]=0,i=1..6), CL);
> simplify(subs(C[4]=-1,seq(C[i]=0,i=1..6), CL));
```

$$\psi_3(t) = const$$

$$\psi_4(t) = const$$

$$\psi_5(t) = const$$

$$\psi_2(t) + \psi_5(t)x_3(t) = const$$

$$
\begin{aligned}
& \frac{1}{2}\,x_1(t)^2x_2(t)^2\psi_5(t)^2 + x_1(t)x_2(t)\psi_5(t)\psi_2(t) + x_1(t)^2x_2(t)\psi_5(t)\psi_3(t) + \frac{1}{2}\,x_1(t)^3x_2(t)\psi_5(t)\psi_4(t) \\
& + \frac{1}{2}\,x_1(t)^2\psi_3(t)^2 + \frac{1}{8}\,x_1(t)^4\psi_4(t)^2 + \frac{1}{2}\,\psi_1(t)^2 + \psi_2(t)x_1(t)\psi_3(t) + \frac{1}{2}\,\psi_2(t)\,x_1(t)^2\psi_4(t) \\
& + \frac{1}{2}\,x_1(t)^3\psi_3(t)\psi_4(t) + \frac{1}{2}\,\psi_2(t)^2 = const
\end{aligned}
$$

One can say that for the Cartan $(2, 3, 5)$ problem we have four trivial first integrals: the Hamiltonian H; and the multipliers ψ_3, ψ_4, ψ_5. Together with the non-trivial integral $x_3\psi_5 + \psi_2$, the problem becomes completely integrable (see [19]).

5 Conservation Laws in the Calculus of Variations

Let us consider the classical problem of the calculus of variations with higher-order derivatives: to minimize an integral functional

$$J[\mathbf{x}(\cdot)] = \int_a^b L(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \ldots, \mathbf{x}^{(r)}(t))\, \mathrm{d}t\,, \tag{18}$$

subject to certain boundary conditions, and where the Lagrangian L depends on the independent variable $t \in \mathbb{R}$, on n dependent variables $\mathbf{x}(t) = [x_1(t),\, x_2(t),\, \cdots,\, x_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$, and its r first derivatives ($\dot{\mathbf{x}}(t) \equiv \mathbf{x}^{(1)}(t)$):

$$\mathbf{x}^{(i)}(t) = \left[\frac{\mathrm{d}^i x_1(t)}{\mathrm{d}t^i}, \frac{\mathrm{d}^i x_2(t)}{\mathrm{d}t^i}, \cdots, \frac{\mathrm{d}^i x_n(t)}{\mathrm{d}t^i}\right]^{\mathrm{T}} \in \mathbb{R}^n\,, \quad i = 1,\, \ldots,\, r\,.$$

It is well known that the problems of the calculus of variations are a particular case of the optimal control problem (1)-(2). The standard technique to write the problem of minimizing (18) as an optimal control problem consists to introduce new functions

$$\mathbf{x}^0 = \mathbf{x},\ \mathbf{x}^1 = \dot{\mathbf{x}},\ \mathbf{x}^2 = \mathbf{x}^{(2)},\ \ldots,\ \mathbf{x}^{r-1} = \mathbf{x}^{(r-1)},\ \text{ and } \mathbf{u} = \mathbf{x}^{(r)}\,.$$

With this notation at hand, the equivalent optimal control problem has rn state variables (x_i^j, $i = 1, \ldots, n$, $j = 0, \ldots, r-1$), and n controls ($\mathbf{u} = \mathbf{x}^{(r)}$):

$$J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] = \int_a^b L(t, \mathbf{x}(t), \mathbf{u}(t))\, \mathrm{d}t \longrightarrow \min\,,$$

$$\begin{cases} \dot{\mathbf{x}}^0(t) & = \mathbf{x}^1(t)\,, \\ \dot{\mathbf{x}}^1(t) & = \mathbf{x}^2(t)\,, \\ & \vdots \\ \dot{\mathbf{x}}^{r-2}(t) & = \mathbf{x}^{r-1}(t)\,, \\ \dot{\mathbf{x}}^{r-1}(t) & = \mathbf{u}(t)\,. \end{cases}$$

Since any problem of the calculus of variations can always be rewritten as an optimal control problem, we can also apply our Maple package to obtain variational symmetries and conservation laws in the classical context of the calculus of variations, and thus recovering the previous investigations of the authors [7]. We recall that for the problems of the calculus of variations there are no abnormal extremals (one can always choose $\psi_0 = -1$). Follow some examples.

Example 19 *(0'08") We begin with a very simple problem of the calculus of variations, where the Lagrangian depends only on one dependent variable ($n = 1$), and where there are no derivatives of higher order than the first one ($r = 1$): $L(t, x, \dot{x}) = t\dot{x}^2$. According with the above mentioned technique of rewriting the problem as an optimal control problem, we write the following definitions in Maple:*

```
> L:=t*v^2; u:=v; phi:=u;
```

$$L := tv^2$$
$$u := v$$
$$\varphi := v$$

Our procedure Symmetry *determine the general infinitesimal generators which define the family of symmetries for the problem of the calculus of variations under consideration:*

```
> Symmetry(L,phi,t,x,u);
```

$$\{T = tC_1,\, X = C_2,\, U = -vC_1,\, \Psi = 0\}$$

The conservation laws corresponding to the computed symmetries, are obtained with the procedure Noether*:*

```
> Noether(L,phi,t,x,u,%);
```

$$C_2\psi(t) - \left(\psi_0 t\,(v(t))^2 + \psi(t)v(t)\right) tC_1 = const$$

Going back to the original notation ($v = \dot{x}$), we can write:

```
> CL:=subs(psi[0]=-1,v(t)=diff(x(t),t),%);
```

$$CL := C_2\psi(t) + \left(t\left(\frac{d}{dt}x(t)\right)^2 - \psi(t)\frac{d}{dt}x(t)\right) tC_1 = const.$$

In this case one can easily use the definition of first integral (a function that is preserved along the extremals of the problem), to verify the validity of the obtained expression. For that we compute the pair $(x(t), \psi(t))$ that satisfies the adjoint system (4) and the maximality condition (5) of the Pontryagin maximum principle (Theorem 1).

```
> H:=-L+psi(t)*phi;
```

$$H := -tv^2 + \psi(t)v$$

```
> {diff(H,u)=0, diff(psi(t),t)=-diff(H,x)};
```

$$\left\{\frac{d}{dt}\psi(t) = 0, -2\,tv + \psi(t) = 0\right\}$$

After substituting $v = \dot{x}$, we obtain the extremals by solving the above system of differential equations:

```
> subs(v=diff(x(t),t),%);
```

$$\left\{\frac{d}{dt}\psi(t) = 0, -2\,t\frac{d}{dt}x(t) + \psi(t) = 0\right\}$$

```
> dsolve(%);
```

$$\left\{\psi(t) = K_2, x(t) = \frac{1}{2}\, K_2\, \ln(t) + K_1\right\}$$

Expression for $x(t)$ coincides with the Euler-Lagrange extremal ([7, Example 5.1]). Substituting the extremals in the conservation law one obtains, as expected, a true proposition:

```
> expand(subs(%,CL));
```

$$C_2 K_2 - \frac{1}{4}\, C_1 {K_2}^2 = const$$

Substituting only $\psi(t)$, one can get the family of conservation laws in the notation of the calculus of variations:

```
> expand(subs(psi(t)=K[2],CL));
```

$$C_2 K_2 + t^2 C_1 \left(\frac{d}{dt} x(t)\right)^2 - t C_1 K_2\, \frac{d}{dt} x(t) = const$$

```
> subs(C[2]*K[2]=0,C[1]=-1,%);
```

$$-t^2 \left(\frac{d}{dt} x(t)\right)^2 + t K_2 \frac{d}{dt} x(t) = const$$

Example 20 (Kepler's problem) *(0'17") We now obtain the conservation laws for Kepler's problem – see [3, p. 217]. In this case the Lagrangian depends on two dependent variables ($n = 2$), without derivatives of higher-order ($r = 1$):*

$$L(t, \mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{K}{\sqrt{q_1^2 + q_2^2}}\,.$$

The family of conservation laws for the problem is easily obtained with our Maple package:

```
> L:= m/2*(v[1]^2+v[2]^2)+K/sqrt(q[1]^2+q[2]^2); x:=[q[1],q[2]]; u:=[v[1],v[2]];
  phi:=[v[1],v[2]];
```

$$\begin{aligned} L &:= \frac{1}{2}\, m \left({v_1}^2 + {v_2}^2\right) + \frac{K}{\sqrt{{q_1}^2 + {q_2}^2}} \\ x &:= [q_1, q_2] \\ u &:= [v_1, v_2] \\ \varphi &:= [v_1, v_2] \end{aligned}$$

```
> Symmetry(L, phi, t, x, u);
```

$$\left\{T = C_3,\, X_1 = -C_1 q_2,\, X_2 = C_1 q_1,\, U_1 = -C_2 v_2 + \frac{(C_1 - C_2)\,\psi_2}{\psi_0 m},\, U_2 = C_2 v_1 - \frac{(C_1 - C_2)\,\psi_1}{\psi_0 m},\right.$$
$$\left.\Psi_1 = -C_1 \psi_2,\, \Psi_2 = C_1 \psi_1\right\}$$

```
> Noether(L, phi, t, x, u, %):
> CL:=subs(psi[0]=-1,%);
```

$$CL := -C_1 q_2(t)\psi_1(t) + C_1 q_1(t)\psi_2(t) - \left(-\frac{1}{2}\,m\left(v_1(t)^2 + v_2(t)^2\right) - \frac{K}{\sqrt{q_1(t)^2 + q_2(t)^2}} + \psi_1(t)v_1(t) + \psi_2(t)v_2(t)\right)C_3 = const$$

To obtain the conservation laws in the format of the calculus of variations, one needs to compute the Pontryagin multipliers $(\psi_1(t), \psi_2(t))$*,*

```
> H:=-L+Vector[row]([psi[1](t), psi[2](t)]).Vector(phi);
```

$$H := -\frac{1}{2}\,m\left({v_1}^2 + {v_2}^2\right) - \frac{K}{\sqrt{{q_1}^2 + {q_2}^2}} + v_1\psi_1(t) + v_2\psi_2(t)$$

```
> solve({diff(H,v[1])=0,diff(H,v[2])=0},{psi[1](t), psi[2](t)});
```

$$\{\psi_1(t) = m v_1, \psi_2(t) = m v_2\}$$

and substitute the expressions, together with $v_1(t) = \dot{x}_1(t)$ *and* $v_2(t) = \dot{x}_2(t)$*:*

```
> expand(subs(%,v[1](t)=v[1],v[2](t)=v[2],v[1]=diff(q[1](t),t),
              v[2]=diff(q[2](t),t),CL));
```

$$-C_1 q_2(t) m \frac{d}{dt}q_1(t) + C_1 q_1(t) m \frac{d}{dt}q_2(t) - \frac{1}{2}\,C_3 m\left(\frac{d}{dt}q_1(t)\right)^2 - \frac{1}{2}\,C_3 m\left(\frac{d}{dt}q_2(t)\right)^2 + \frac{C_3 K}{\sqrt{q_1(t)^2 + q_2(t)^2}} = const$$

This is the conservation law in [7, Example 5.2].

Example 21 *(6'42") Let us see an example of the calculus of variations whose Lagrangian depends on two functions* $(n = 2)$ *and higher-order derivatives* $(r = 2)$*:*

$$L(t, \mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \dot{x}_1^2 + \ddot{x}_2^2\,.$$

We write the problem in the optimal control terminology, and make use of our Maple procedure Symmetry *to compute the symmetries:*

```
> L:=v[1]^2+a[2]^2; xx:=[x[1],x[2],v[1],v[2]]; u:=[a[1],a[2]];
  phi:=[v[1],v[2],a[1],a[2]];
```

$$\begin{aligned} L &:= {v_1}^2 + {a_2}^2 \\ xx &:= [x_1, x_2, v_1, v_2] \\ u &:= [a_1, a_2] \\ \varphi &:= [v_1, v_2, a_1, a_2] \end{aligned}$$

```
> Symmetry(L, phi, t, xx, u);
```

$$\Big\{T = 2\,C_3 t + C_4,\, X_1 = C_3 x_1 + C_5,\, X_2 = tC_1 + 3\,C_3 x_2 + C_6,\, X_3 = -C_3 v_1,\, X_4 = C_1 + C_3 v_2,\, U_1 = -3\,C_3 a_1 - 2\,\psi_0 C_2 a_2 - C_2\psi_4,\, U_2 = -C_3 a_2 + C_2\psi_3,\, \Psi_1 = -C_3\psi_1,\, \Psi_2 = -3\,C_3\psi_2,\, \Psi_3 = C_3\psi_3,\, \Psi_4 = -C_3\psi_4\Big\}$$

We choose, in the conservation law returned by our Maple procedure Noether, $\psi_0 = -1$, *and then go back to the calculus of variations notation:* $v_1(t) = \dot{x}_1(t)$, $v_2(t) = \dot{x}_2(t)$, $a_1(t) = \ddot{x}_1(t)$, *and* $a_2(t) = \ddot{x}_2(t)$*:*

```
> Noether(L, phi, t, xx, u, %):
> CL:=subs(psi[0]=-1, v[1](t)=diff(x[1](t),t), v[2](t)=diff(x[2](t),t),
  a[1](t)=diff(x[1](t),t$2), a[2](t)=diff(x[2](t),t$2),%);
```

$$CL := (C_3 x_1(t) + C_5)\,\psi_1(t) + (tC_1 + 3\,C_3 x_2(t) + C_6)\,\psi_2(t) - C_3\left(\frac{d}{dt}x_1(t)\right)\psi_3(t) + \left(C_1 + C_3\frac{d}{dt}x_2(t)\right)\psi_4(t) - \left(-\left(\frac{d}{dt}x_1(t)\right)^2 - \left(\frac{d^2}{dt^2}x_2(t)\right)^2 + \psi_1(t)\frac{d}{dt}x_1(t) + \psi_2(t)\frac{d}{dt}x_2(t) + \psi_3(t)\frac{d^2}{dt^2}x_1(t) + \psi_4(t)\frac{d^2}{dt^2}x_2(t)\right)(2\,C_3 t + C_4) = const$$

Similarly to Example 19, we can also compute through Maple the extremals and verify, by definition, the validity of the obtained family of conservation laws.

```
> vpsi:=Vector[row]([psi[1](t), psi[2](t), psi[3](t), psi[4](t)]):
> H:=-L+vpsi.Vector(phi);
```

$$H := -{v_1}^2 - {a_2}^2 + v_1\psi_1(t) + v_2\psi_2(t) + a_1\psi_3(t) + a_2\psi_4(t)$$

```
> {diff(H,u[1])=0, diff(H,u[2])=0, diff(vpsi[1],t)=-diff(H,xx[1]),
   diff(vpsi[2],t)=-diff(H,xx[2]), diff(vpsi[3],t)=-diff(H,xx[3]),
   diff(vpsi[4],t)=-diff(H,xx[4])}:
> subs(v[1]=diff(x[1](t),t), a[2]=diff(x[2](t),t$2), %);
```

$$\left\{\frac{d}{dt}\psi_3(t) = 2\,\frac{d}{dt}x_1(t) - \psi_1(t), \frac{d}{dt}\psi_2(t) = 0, \frac{d}{dt}\psi_4(t) = -\psi_2(t), \psi_3(t) = 0, \frac{d}{dt}\psi_1(t) = 0, -2\,\frac{d^2}{dt^2}x_2(t) + \psi_4(t) = 0\right\}$$

Solving the above system of equations, that result from the maximality condition and adjoint system,

```
> dsolve(%);
```

$$\Big\{x_1(t) = \frac{1}{2}\,K_6\,t + K_4,\; x_2(t) = -\frac{1}{12}\,K_5\,t^3 + \frac{1}{4}\,K_3\,t^2 + K_1\,t + K_2, \psi_3(t) = 0,$$
$$\psi_1(t) = K_6,\; \psi_4(t) = -K_5\,t + K_3,\; \psi_2(t) = K_5\Big\}$$

we obtain the extremals (the extremal state trajectories $x_1(t)$ and $x_2(t)$ are the same as the ones obtained in [7, Example 5.3], by solving the Euler-Lagrange necessary optimality condition) that, substituted in the conservation law,

```
> expand(subs(%,CL));
```

$$K_6\,C_3K_4 + K_6\,C_5 + 3\,K_5\,C_3K_2 + K_5\,C_6 + C_1K_3 + C_3K_1\,K_3 - \frac{1}{4}\,{K_6}^2C_4$$
$$-\frac{1}{4}\,{K_3}^2C_4 - K_5\,K_1\,C_4 = const$$

conduces to a true proposition (constant equal constant). Finally, substituting only the Pontryagin multipliers,

```
> subs({psi[1](t)=K[6], psi[3](t)=0, psi[4](t)=-K[5]*t+K[3], psi[2](t)=K[5]}, CL);
```

$$(C_3x_1(t) + C_5)\,K_6 + (C_1t + 3\,C_3x_2(t) + C_6)\,K_5 + \left(C_1 + C_3\frac{d}{dt}x_2(t)\right)(-K_5\,t + K_3)$$
$$-\left(-\left(\frac{d}{dt}x_1(t)\right)^2 - \left(\frac{d^2}{dt^2}x_2(t)\right)^2 + K_6\,\frac{d}{dt}x_1(t) + K_5\,\frac{d}{dt}x_2(t)\right.$$
$$\left. + (-K_5\,t + K_3)\,\frac{d^2}{dt^2}x_2(t)\right)(2\,C_3t + C_4) = const$$

the conservation law takes the form of a differential equation of less order than the one obtained in [7, Example 5.3] (the Hamiltonian approach is here more suitable than the Lagrangian one).

Example 22 (Emden-Fowler) *(0'01") Given the variational problem of Emden-Fowler [3, p. 220], defined by the Lagrangian*

```
> L:= t^2/2*(v^2-x^6/3);
```

$$L := \frac{1}{2}\,t^2\left(v^2 - \frac{1}{3}\,x^6\right)$$

we are interested to find, following our methodology, the conservation laws for the problem.

```
> Symmetry(L, v, t, x, v);
```

$$\{T = -2\,tC_1,\; X = C_1x,\; U = 3\,C_1v,\; \Psi = -\psi\,C_1\}$$

```
> Noether(L,v,t,x,v,%):
> CL:=subs(psi[0]=-1,%);
```

$$CL := C_1 x(t)\psi(t) + 2\left(-\frac{1}{2}t^2\left(v(t)^2 - \frac{1}{3}x(t)^6\right) + \psi(t)v(t)\right)tC_1 = const$$

The expression for the $\psi(t)$ comes from the stationary condition.

```
> H:=-L+psi(t)*v;
```

$$H := -\frac{1}{2}t^2\left(v^2 - \frac{1}{3}x^6\right) + \psi(t)v$$

```
> solve(diff(H,v)=0,{psi(t)});
```

$$\{\psi(t) = t^2 v\}$$

```
> subs(%,v(t)=diff(x(t),t),v=diff(x(t),t),CL): expand(%);
```

$$C_1 x(t)t^2\frac{d}{dt}x(t) + t^3 C_1\left(\frac{d}{dt}x(t)\right)^2 + \frac{1}{3}t^3 C_1 x(t)^6 = const$$

Fixing $C_1 = 3$,

```
> subs(C[1]=3,%);
```

$$3\,x(t)t^2\frac{d}{dt}x(t) + 3\,t^3\left(\frac{d}{dt}x(t)\right)^2 + t^3 x(t)^6 = const$$

we obtain the same conservation law as the one obtained in [7, Example 5.4], with the methods of the calculus of variations.

Example 23 (Thomas-Fermi) *(0'01") We consider the problem of Thomas-Fermi [3, p. 220], showing an example of a problem of the calculus of variations which does not admit variational symmetries.*

```
> L:=1/2*v^2+2/5*x^(5/2)/sqrt(t);
```

$$L := \frac{1}{2}v^2 + \frac{2}{5}\frac{x^{\frac{5}{2}}}{\sqrt{t}}$$

```
> Symmetry(L, v, t, x, v);
```

$$\{T = 0,\, X = 0,\, U = 0,\, \Psi = 0\}$$

Our Maple function Symmetry *returns, in this case, vanishing generators. As explained in §3, this means that the problem does not admit symmetries.*

Example 24 (Damped Harmonic Oscillator) *(0'02") We consider a harmonic oscillator with restoring force $-kx$, emersed in a liquid in such a way that the motion of the mass m is damped by a force proportional to its velocity. Using Newton's second law one obtains, as the equation of motion, the Euler-Lagrange differential equation associated with the following Lagrangian [11, pp. 432–434]:*

```
> L:=1/2*(m*v^2-k*x^2)*exp((a/m)*t);
```

$$L := \frac{1}{2}\left(mv^2 - kx^2\right) e^{\frac{at}{m}}$$

In order to find the conservation laws, we first obtain, as usual, the generators which define the symmetries of the problem.

```
> simplify(Symmetry(L,v,t,x,v));
```

$$\left\{T = -\frac{2\,mC_1}{a},\ X = C_1 x,\ U = C_1 v,\ \Psi = -\psi\, C_1\right\}$$

```
> Noether(L,v,t,x,v,%):
> CL:=subs(psi[0]=-1,%);
```

$$CL := C_1 x(t)\psi(t) + 2\left(-\frac{1}{2}\left(m\,v(t)^2 - k\,x(t)^2\right) e^{\frac{at}{m}} + \psi(t)v(t)\right) mC_1 a^{-1} = const$$

The value for $\psi(t)$ is easily determined, and we can write the obtained family of conservation laws in the language of the calculus of variations.

```
> H:=-L+psi(t)*v;
```

$$H := -\frac{1}{2}\left(mv^2 - kx^2\right) e^{\frac{at}{m}} + \psi(t)v$$

```
> solve(diff(H,v)=0,{psi(t)});
```

$$\left\{\psi(t) = m\,v\,e^{\frac{at}{m}}\right\}$$

```
> simplify(subs(%,v(t)=diff(x(t),t),v=diff(x(t),t),CL));
```

$$C_1 m e^{\frac{at}{m}}\left(x(t)\left(\frac{d}{dt}x(t)\right)a + m\left(\frac{d}{dt}x(t)\right)^2 + kx(t)^2\right)a^{-1} = const$$

Choosing an appropriate value to the constant C_1

```
> subs(C[1]=-a/(2*m),%);
```

$$-\frac{1}{2}\,e^{\frac{at}{m}}\left(x(t)\left(\frac{d}{dt}x(t)\right)a + m\left(\frac{d}{dt}x(t)\right)^2 + kx(t)^2\right) = const$$

we obtain the conservation law in [11, Ch. 7, Example 1.10].

6 The Maple package

The procedures *Symmetry* and *Noether*, described in the previous sections, have been implemented for the computer algebra system Maple (version 9.5).

Symmetry computes the infinitesimal generators which define the symmetries of the optimal control problem specified in the input. As explained in sections 2 and 3, this procedure involves the solution of a system of partial differential equations. We have used the Maple solver *pdsolve*, trying to separate the variables by sum.

Output:

- set of infinitesimal generators.

Syntax:

- Symmetry(L, φ, t, x, u, [`all`])

Input:

L - expression of the Lagrangian;

φ - expression or list of expressions of the velocity vector $\boldsymbol{\varphi}$ which defines the control system;

t - name of the independent variable;

x - name or list of names of the state variables;

u - name or list of names of the control variables;

`all` - This is an optional parameter. When *all* is given in the last argument of the procedure *Symmetry*, the output presents all the constants given by the Maple command *pdsolve*. By default, that is, without optional *all*, we eliminate redundant constants. This is done by our Maple procedure *reduzConst*. This is a technical routine, and thus not provided here. Essentially, the procedure transforms in one constant each sum of constants not repeated in the conservation law. The interested reader can find the Maple file with its definition, together with the *Symmetry* and *Noether* code, that constitute our Maple package, at http://www.mat.ua.pt/delfim/maple.htm.

```
Symmetry:=proc(L::algebraic, phi::{algebraic, list(algebraic)}, t::name,
                                    x0::{name,list(name)}, u0::{name,list(name)})
  local n,m, xx, i, vX, vPSI, vU, vv, lpsi, H, eqd, syseqd, sol, conjGerad, lphi;
  unprotect(Psi); unassign('T'); unassign('X'); unassign('U'); unassign('Psi');
  unassign('psi');
  n:=nops(x0); m:=nops(u0);
  if n>1 then lphi:=phi;lpsi:=[seq(psi[i],i=1..n)];
  else lphi:=[phi]; lpsi:=[psi]; fi;
```

```
  xx:=op(x0),op(u0),op(lpsi); vv:=Vector([seq(v||i,i=1..2*n+m)]);
  if n>1 then vX:=Vector([seq(X[i](t,xx), i=1..n)]);
  else vX:=Vector([X(t,xx)]); fi;
  if n>1 then vPSI:=Vector([seq(PSI[i](t,xx),i=1..n)]);
  else vPSI:=Vector([PSI(t,xx)]); fi;
  if m>1 then vU:=Vector([seq(U[i](t,xx), i=1..m)]);
  else vU:=Vector([U(t,xx)]); fi;
  H:=psi[0]*L+Vector[row](lphi).Vector(lpsi);
  eqd:=diff(H,t)*T(t,xx) +Vector[row]([seq(diff(H,i),i=x0)]).vX+Vector[row]([seq(
       diff(H,i),i=u0)]).vU+Vector[row]([seq(diff(H,xx[i]),i=n+m+1..n+m+n)]).vPSI
       -LinearAlgebra[Transpose](vPSI).vv[1..n]-Vector[row](lpsi).(map(diff,vX,t)
                          +Matrix([seq(map(diff,vX,i),i=xx)]).vv)+H*(diff(T(t,xx),t)
                                     +Vector[row]([seq(diff(T(t,xx),i),i=xx)]).vv);
  eqd:=expand(eqd);  eqd:=collect(eqd, convert(vv,'list'), distributed);
  syseqd:={coeffs(eqd, convert(vv,'list'))}:
  conjGerad:={T(t,xx)}union convert(vX,'set') union convert(vU,'set')
                                                     union convert(vPSI,'set');
  sol:=pdsolve(syseqd, conjGerad, HINT=`+`);
  sol:=subs(map(i->i=op(0,i),conjGerad),sol); sol:=subs(PSI='Psi',sol);
  if nargs<6 or args[6]<>`all` then sol:=reduzConst(sol); fi;
  return sol;
end proc:
```

Noether given the infinitesimal generators which define a symmetry, computes the conservation law for the optimal control problem, according with Theorem 7 (Noether's theorem).

Output:

- conservation law.

Syntax:

- Noether(L, φ, t, x, u, S)

Input:

- L - expression of the Lagrangian;
- φ - expression or list of expressions of the velocity vector $\boldsymbol{\varphi}$ which defines the control system;
- t - name of the independent variable;
- x - name or list of names of the state variables;
- u - name or list of names of the control variables;
- S - set of infinitesimal generators (output of procedure *Symmetry*).

```
Noether:=proc(L::algebraic, phi::{algebraic, list(algebraic)}, t::name,
                            x0::{name,list(name)}, u0::{name,list(name)}, S::set)
  local n, xx, i, vX, vpsi, lpsi,  H, CL, lphi;
  unassign('T'); unassign('X'); unassign('psi');
  n:=nops(x0);
  if n>1 then lphi:=phi; lpsi:=[seq(psi[i],i=1..n)];
  else lpsi:=[psi]; lphi:=[phi]; fi;
  xx:=op(x0),op(u0),op(lpsi);
  vpsi:=Vector[row](lpsi);
  if n>1 then vX:=Vector([seq(X[i], i=1..n)]); else vX:=Vector([X]); fi;
  H:=psi_0*L+vpsi.Vector(lphi);
  CL:=vpsi.vX-H*T=const;
  CL:=eval(CL, S);
  CL:=subs({map(i->i=i(t),[xx])[]},CL); CL:=subs(psi_0=psi[0],CL);
  return CL;
end proc:
```

7 Concluding Remarks

Computer Algebra Systems are particularly suitable to handle the problem of determining symmetries and conservation laws in optimal control, because theory requires calculations that tend to be tedious even for very simple problems with a linear control system. Maple can perform these computations in a reliable way. We illustrate our package by a large number of examples, that range from simple problems in the calculus of variations to optimal control problems of which the integrability was only very recently shown by means of the conservation laws. Now these computations can be made in a completely automatic way and without any physical insights.

In mechanics, and calculus of variations, it is well known how to use the conservation laws to reduce the order of the problems and, with a sufficiently large number of independent conservation laws, one can even integrate and solve the problems completely. Like Noether's theorem, also the classical reduction theory can be extended to the more general setting of optimal control [5, 13]. However, the reduction theory in optimal control is an area not yet completed. More theoretical results are needed in order to be possible to automatize the whole process, from the computation of symmetries to the maximum reduction of the problems.

As the examples show, the computing times increase exponentially with the dimension of the control system (with the number of state variables). This is well illustrated with the problems of sub-Riemannian geometry: problem $(2, 3)$ (Example 17), with three state variables, requires a total computing time of one minute; problem $(2, 3, 5)$ (Example 18), with five state variables, has required us thirty minutes. To tackle more complex problems a new approach is needed. This is also under study and will be addressed elsewhere. For example, integrability of the $(2, 3, 5, 8)$ problem of sub-Riemannian geometry, a problem with eight state variables, is currently an open question. This problem is out of the scope of the present Maple package: eleven hours of computing time were not enough for us to determine the symmetries of the problem.

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