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CONTROL PROBLEM IN PASSIVE TRACER ADVECTION BY POINT VORTEX FLOW: A CASE STUDY

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Abstract. *Vortex dynamics and passive tracers in vortex-dominated flows form a vast area of research that continues to attract the attention of numerous studies. Among these studies, it has emerged in recent times a special interest in the use of control theory applied to vortex dynamics. Point vortices are singular solutions of the two-dimensional incompressible Euler equations. These solutions correspond to the limiting case where the vorticity is completely concentrated on a finite number of spatial points, each with a prescribed strength/circulation. By definition, a passive tracer is a point vortex with zero circulation. We are concerned with the dynamics of a passive tracer advected by two-dimensional point vortex flow. More precisely, we want to drive a passive particle from an initial starting point to a final terminal point, both given a priori, in a given finite time. The flow is originated by the displacement of* N *viscous point vortices. More precisely, we look for the optimal trajectories that minimize the objective function that correspond to the energy expended in the control of the trajectories. The restrictions are essentially due to the ordinary differential equations that govern the displacement of the passive particle around the viscous point vortices.*

1 INTRODUCTION

Vortex dynamics and passive tracers in vortex-dominated flows form a vast area of research that continues to attract the attention of numerous studies. Point vortex are mathematical models used to describe the dynamic of vortex-dominated flows. These models are based on a low dimensional description of the flow features [1]. Vortex dynamics, based on point vortex models, have been employed in many science and engineering areas like geophysics, turbulence, superfluids or hydrodynamic [2, 3, 4, 5, 6]. The solution of these vortex dynamics models is usually obtained with low computational costs. Whereas, the solution of partial differential equations in realistic problems requires high CPU time and large memory storage. This makes these models very attractive, especially in flow control problems [7].

It has emerged in recent years a special interest in the use of control theory applied to vortex dynamics. In most of the control problems, concerning realistic flows, the solution is achieved by means of simplified models such as point vortex [1]. In consequence, there is a special interest in the use of control methods applied to vortex dynamics, namely in the fields of geophysical fluid dynamics, aeronautics and hydrodynamics [7].

In the context of hydrodynamics, the fish-like locomotion is an application of point vortex that have received some attention in the last years. This is mostly due to the development of autonomous underwater vehicles for the collection of data concerning the multiple oceanic phenomena [8]. In some approaches the robotic fish locomotion is modelled through a point mass in vector fields defined by the vortex dynamics. In [9], the displacement is achieved by the constant generation of periodic and predefined vortices. In [10], the control action is exercised by the generation of one vortex.

Point vortex dynamics is an area of mathematical physics that has served as a classical mathematics *playground* [11]. Over the times, many different methods from pure and applied mathematics have been used in this area. From the control point of view, several techniques have been also applied. Somme of there *had-oc* and other based on solid mathematical foundations [7].

There is two main class of control methods: direct and indirect. Generally speaking, the direct approach consist to firstly discretize the problem and after that optimize, while the indirect approach, firstly optimize and after discretize. Direct methods discretize the problem relatively to the time in order to get a Non-Linear Programming (NLP) that can be solved by an optimization method like Interior Point. These methods are generally handy for singular or constrained arcs of the trajectory, but their accuracy can be affected by the discretization [12, 13]. Indirect methods use Pontryagin's Maximum Principle to derive optimal conditions, where it is necessary to maximization the Hamiltonian, which can be achieved by collocation or shooting methods [14]. Indirect methods are fast and accurate, but they are also sensitive to the starting guess of the adjoint problem.

This work is concerned with the dynamics of a passive particles advected by a two-dimensional point vortex flow. A passive particle is small enough not to perturb the velocity field, but also large enough not to perform a Brownian motion. Particles of this type are the tracers used for flow visualization in fluid mechanics experiments [15]. We consider also that the passive particle have the same density of the fluid in which it is embedded.

We want to drive a passive particle from an initial starting point to a final terminal point, both given *a priori*, in a given finite time. The flow is originated by the displacement of a certain number, say N, of point vortices. This problem as some similarities with the fish-like locomotion problem. Here the vortex dynamics is governed by N point vortices and the control is due to the possibility of impulsion in any direction of the two dimensional plane. Of course we want minimize the total amount of energy spent in the impulsion. This issue can also be seen as part of the set of general open control problems proposed by Protas [7].

The displacement of the passive particle is then transformed in a control problem that we solve by a direct approach. The time disposable to preform the displacement is divided in a fixed number n of subinterval, where the control variables are constant. The discretized problem is solved numerically by mean of a single shooting method. In each subinterval the vortex dynamics is integrated by the fourth order Runge-Kutta method.

Our approach is different from the usual ones, that are based on the indirect approach. Normally the Pontryagin's Maximum Principle is used to derive the necessary conditions of optimality of the problem. This results in a problem with hight complexity, hose resolution needs some simplifying assumptions. On of these simplifications is to consider that the vortex dynamic is induced just by two point vortices [8].

In these paper we consider that the passive particle moves in a two dimensional flow whose dynamic is given, at any time interval, by N point vortices. We consider four different problems, each one corresponding to a different value of N, ranged from one to four. We formulate the non linear programme (NLP) corresponding to each case and we show numerically that there is an optimal control for each one. The solution of the NLP is achieved by mean of the Matlab Optimization Toolbox^{TM}.

In Section 2 we introduce the equations that give the motion of passive particles advected by N point vortices in the infinite real plane. After that, in Section 3, we formulate the control problems. Section 4 is devoted to the numerical solution of these control problems. We start by a passive particle moved by a single point vortex (Subsection 4.1.1) and terminate with a movement induced by four point vortex (Subsection 4.2.3). This paper is closed in Section 5 with some conclusions.

2 TWO-DIMENSIONAL EULER EQUATION AND POINT VORTEX

The two-dimensional incompressible Euler equations are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\nabla p \,, \tag{1}$$

and

$$\nabla \cdot \mathbf{u} = 0 \,, \tag{2}$$

plus initial and boundary conditions, wherein $\mathbf{u} = (u_1, u_2)\,, \quad u_i = u_i(X, t) \quad (i = 1, 2)$ is the two-dimensional incompressible velocity field, $X = (x, y) \in \mathbb{R}^2$ the space coordinates, t is the time variable, $\nabla = (\partial_x, \partial_y)$ is the gradient, and p is the pressure.

Due to the incompressible condition in two spatial dimensions, i.e., $\nabla \cdot \mathbf{u} = 0$, one can express the velocity field $\mathbf{u}$ in term of the so-called stream-function Ψ:

$$\mathbf{u} = (u_1, u_2) = (\partial_y \Psi, -\partial_x \Psi) \ . \tag{3}$$

Introducing the vorticity vector (consider $\mathbf{u}$ in $\mathbb{R}^3$, where the 3rd component is zero):

$$\mathbf{w} = \nabla \times \mathbf{u} = (0, 0, \partial_x u_2 - \partial_y u_1)$$

the 2D scalar vorticity, given by

$$\omega = (\mathbf{w})_3 = \partial_x u_2 - \partial_y u_1 \,, \tag{4}$$

is linked with the stream-function through the Poisson equation

$$\nabla^2 \Psi = -\omega \,. \tag{5}$$

Taking the curl in both sides of 2D Euler equation leads us to the vorticity formalism of the Euler equations [16]

$$\partial_t \omega + (u \cdot \nabla)\, \omega \equiv \frac{d\omega}{dt} = 0 \,. \tag{6}$$

The equation (6) enables us to get the time evolution of the vorticity. Once we have the vorticity, the stream-function is computed through the solution of the Poisson equation (5). In turn, the velocity field is obtained by equation (3). This is the typical procedure used to solve the Euler equation.

Effectively, it can be shown (see [17]) that the solution of the Poisson equation (5), in the whole plane, is given by

$$\Psi = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \ln |X - X'|\, \omega(X')\, dX' \,, \tag{7}$$

and that **u** is equal to the convolution between the kernel $K(\cdot,\cdot)$ and the given vorticity:

$$\mathbf{u} = (\partial_y \Psi, -\partial_x \Psi) = K * \omega \,, \tag{8}$$

with

$$K(x,y) = \frac{1}{2\pi} \left(\frac{y}{x^2+y^2} \,,\; -\frac{x}{x^2+y^2} \right) . \tag{9}$$

The point vortices correspond to the particular case where the vorticity is given by the weighted sum

$$\omega = \omega\,(t, X) = \sum_{\alpha=1}^{N} k_\alpha \, \delta\,(X - X_\alpha\,(t)) \,, \tag{10}$$

where N is equal to the number of vortices and $\delta(\cdot)$ is the $\delta-$Dirac function. The quantity k_α is the circulation of the vortex α ($\alpha = 1, 2, \cdots, N$) that, at time t, is located in $X_\alpha(t) = (x_\alpha(t), y_\alpha(t))$.

The vortex equations are

$$\begin{aligned} \frac{dx_\alpha}{dt} &= -\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{k_\beta}{2\pi} \frac{y_\alpha - y_\beta}{r^2_{\alpha\beta}} \,, \\ \frac{dy_\alpha}{dt} &= \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{k_\beta}{2\pi} \frac{x_\alpha - x_\beta}{r^2_{\alpha\beta}} \,, \end{aligned} \tag{11}$$

supplemented by appropriate initial conditions. Here, $r^2_{\alpha\beta} = (x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2$.

The equations (11) can be written using the complex variables

$$\frac{dz^*_\alpha}{dt} = \frac{1}{2\pi i} \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{k_\beta}{z_\alpha - z_\beta} \quad (\alpha = 1, 2, \ldots, N) \,, \tag{12}$$

whereas $z_\alpha = x_\alpha + y_\alpha i$ $(i^2 = -1)$ and z^*_α its complex conjugate.

A passive particle is by definition a point vortex with circulation $k = 0$. Thus, the dynamics of a system with P passive particles advected by N point vortices is given by (12) together with the equations for the passive particles

$$\frac{dz_\alpha^*}{dt} = \frac{1}{2\pi i} \sum_{\beta=1}^{N} \frac{k_\beta}{z_\alpha - z_\beta} \quad (\alpha = N+1, N+2, \ldots, N+P)\,, \tag{13}$$

with the respective initial conditions.

3 STATEMENT OF THE CONTROL PROBLEM

The abstract formulation of the problem we want to address can be stated as follows. Consider the state vector ($M = N + P$)

$$\mathbf{X}(t) = [x_1(t)\; y_1(t)\; x_2(t)\; y_2(t)\; \cdots\; x_M(t)\; y_M(t)]^T \in \mathbb{R}^{2M}\,, \tag{14}$$

and the evolution systems (12)-(13) concisely written as

$$\frac{d\mathbf{X}}{dt} = \mathbf{f}(\mathbf{X})\,, \quad \mathbf{X}(0) = \mathbf{X}_0\,, \tag{15}$$

where $\boldsymbol{f} : \mathbb{R}^{2M} \to \mathbb{R}^{2M}$ is the function describing the advection velocities of the vortices and particles due to the induction of all the vortices.

If we denote $\mathbf{U} :\; [0, +\infty[\to \mathbb{R}^m$ a time dependent input control with m degree of freedom, the system (15) can be reconfigured to

$$\frac{d\mathbf{X}}{dt} = \mathbf{f}(\mathbf{X}) + \mathbf{b}(\mathbf{X})\mathbf{U}(t)\,, \quad \mathbf{X}(0) = \mathbf{X}_0\,. \tag{16}$$

where $\mathbf{b}(\mathbf{X})$: $\mathbb{R}^m \to \mathbb{R}^{2M}$ is the control operator that describes the control action, $\mathbf{U}(\cdot)$, in the system dynamics.

The control problem can then be stated as follows: given the initial state $\mathbf{X}_0$ of the system and the prescribed $\mathbf{X}_T = \mathbf{X}(T)$ terminal state, determine $\mathbf{U}(\cdot)$ that drives $\mathbf{X}_0$ to $\mathbf{X}_T$ during the time interval $[0, T]$ and minimizing, for instance, T, or some cost function, etc.

4 SOLVING THE CONTROL PROBLEM BY DIRECT APPROACH

In this Section, we present the numerical approach used to solve the control problem presented above. Our direct approach consists in discretizing the problem and solving it using an optimization method. In this way, the control function $\mathbf{U}(\cdot)$ is replaced by n control variables u_0, $u_1, \cdots, u_{n-1}$. The calculations were performed in Matlab®, with the nonlinear programming solver `fmincon`. This solver provides some constrained optimization algorithms, such as the Interior Point or the Active-Set (see [18] and [19], for instance). We begin, in Section 4.1, by solving this problem for the case of a single passive particle in a single vortex flow and, after that, in Section 4.2, we address the cases involving up to four vortices. We would like to point out that the cases N = 2 and N = 3 correspond to integrable point vortex dynamics, whereas N = 4 (or higher) corresponds to chaotic point vortex dynamics [15, 20].

4.1 Flow created by one single vortex ($N = 1$) and one particle ($P = 1$).

In this Section, we address the case of a flow created by one single vortex. In subsection 4.1.1, we introduce the corresponding discretized optimization problem. For the case of a particular complex control variable, we show that the problem formulated in Section 3 has an analytical solution. For the general case we show the numerical solutions corresponding to a different number of control variables ($n = 1, 2, 3$ and 4).

4.1.1 Optimization Problem.

From a practical point of view, the control problem introduced in Section 3 can be illustrated for the case of one single passive particle ($P = 1$) which moves thanks to the presence a single vortex ($N = 1$) as follow:

$$\begin{array}{lll} (\mathcal{P}) & \text{Minimize} & \int_0^T |u(t)|^2\, dt \\ & \text{subject to} & \dot{z}^\star = \frac{1}{2\pi i} \times \frac{k}{z} + u \\ & & z(0) = z_0 \\ & & z(T) = z_f \\ & & |u| \leq u_{\max} \end{array}$$

with $u \in \mathbb{C}$, and $z_0\, , z_f \in \mathbb{C}$, $T > 0$ and $u_{\max} > 0$ given.

In this optimization problem, the objective (cost) function represents, for instance, the energy spent by the control made by the passive particle along the displacement from z_0 to z_f. The first restriction corresponds to the state equation that governs the position z of the particle as a function of the time. The control function u is introduced in this equation in order to move the particle from the starting point z_0 to the end point z_f in a fixed value of time T. Theses two points are previously defined, as well as the time T available for travel from z_0 to z_t. Additionally, in the fourth restriction, we impose that the control module is not greater than a prescrived value $u_{\max}$.

To address this problem, we proceed to the discretization of the control function. We replace $u(\cdot)$ by n (discrete) variables defined as ($t_0 = 0,\ t_n = T$)

$$\begin{array}{lll} u(t) = u_0 & \text{if} & t_0 \leq t < t_1\,, \\ u(t) = u_1 & \text{if} & t_1 \leq t < t_2\,, \\ u(t) = u_2 & \text{if} & t_2 \leq t < t_3\,, \\ & \vdots & \\ u(t) = u_{n-1} & \text{if} & t_{n-1} \leq t \leq t_n\,. \end{array}$$

Thus, each variable u_i ($i = 1, 2, \cdots, n$) corresponds to a constant value of the control exercised in the sub-interval $[t_{i-1}, t_i)$. Each of these subintervals has the constant length $\Delta t = (t_n - t_0)/n$.

The discretization of the objective function by the rectangle method lead to the approximation

$$\int_0^T |u|^2\, dt \approx \Delta t \left(|u_0|^2 + |u_1|^2 + |u_2|^2 + \cdots + |u_{n-1}|^2\right)\,. \tag{17}$$

The control problem ($\mathcal{P}$) is then replaced by its discretized version:

$$\begin{array}{lll} (\mathcal{DP}_n) & \text{Minimize} & \Delta t \sum_{k=0}^{n-1} |u_k|^2 \\ & \text{subject to} & \dot{z}^\star = \frac{k}{2\pi i}\frac{1}{z} + u_0, \quad z(0) = z_0, \quad |u_0| \leq u_{\max}, \quad t_0 \leq t < t_1 \\ & & \dot{z}^\star = \frac{k}{2\pi i}\frac{1}{z} + u_1, \quad z(t_1) = z_{t_1}, \quad |u_2| \leq u_{\max}, \quad t_1 \leq t < t_2 \\ & & \vdots \\ & & \dot{z}^\star = \frac{k}{2\pi i}\frac{1}{z} + u_{n-1}, \quad z(t_{n-1}) = z_{t_{n-1}}, \quad |u_{n-1}| \leq u_{\max}, \quad t_{n-1} \leq t < t_n \\ & & z(t_n) = z_f \end{array}$$

The case $\mathbf{n = 1}$ with $\mathbf{u_0} \in \mathbb{C}$ and $\boldsymbol{b(z) = \alpha/z}$. We begin by studying the case of a single vertex, located at the origin, for the particular situation, where the operator control $\boldsymbol{b}(\cdot)$ in (16) is of the form $\boldsymbol{b}(z) = \boldsymbol{\alpha/z}$ ($\alpha \in \mathbb{R} \setminus \{0\}$) there is an analytical solution of the optima control.

Effectively, the state equation becomes

$$\dot{z}^\star = \frac{k}{2\pi i}\frac{1}{z} + \frac{\alpha}{z}u_0\,, \tag{18}$$

with $k \in \mathbb{R}$, $u_0 = u + iv \in \mathbb{C}$. We want to find u and v in order to drive $z(0) = \rho_0 e^{i\,\theta_0}$ to $z(T) = \rho_T e^{i\,\theta_T}$, in time T. For this, we start by writing $z(\cdot)$ in polar coordinates, that is, $z(t) = \rho(t)e^{i\,\theta(t)}$. Substituting this transformation into (18) and matching the real and imaginary parts to the resulting equation, we get

$$\left\{\begin{array}{lcl} \dot{\rho} & = & \frac{\alpha}{\rho}\,u, \\ \dot{\theta} & = & \frac{k}{2\pi\rho^2} - \frac{\alpha}{\rho^2}v\,. \end{array}\right.$$

A straightforward integration allows us to obtain ($\rho(0) = \rho_0$ and $\theta(0) = \theta_0$)

$$\left\{\begin{array}{lcl} \rho(t) & = & \sqrt{2\alpha ut + \rho_0^2}\,, \\ \theta(t) & = & \theta_0 + \frac{k - 2\pi\alpha v}{4\pi\alpha u}\ln\left(1 + \frac{2\alpha ut}{\rho_0^2}\right)\,. \end{array}\right.$$

Finally, the determination of the control $u_0 = u + iv$ is made thanks to terminal conditions $\rho(T) = \rho_T$ and $\theta(T) = \theta_T$, getting then

$$\left\{\begin{array}{lcl} u & = & \frac{\rho_T^2 - \rho_o^2}{2\alpha T}\,, \\ v & = & \frac{k}{2\pi\alpha} - \frac{1}{2\alpha T}\left(\rho_T^2 - \rho_o^2\right)\left(\theta_T - \theta_0\right)\frac{1}{\ln(\rho_T) - \ln(\rho_0)}\,. \end{array}\right.$$

In this particular case there is an analytical solution. Next, we present the numerical resolution for the single vortex problem, located at the origin.

The case $\mathbf{n = 1}$ with $\mathbf{u_0} \in \mathbb{C}$. The general case of a single vertex, located at the origin, is solved numerically. We seek for a number $u_0 \in \mathbb{C}$ that drives the passive particle from z_0 to z_f in exactly T units of time and minimizes the energy. This problem is formulated as follows:

$$\begin{array}{lll} (\mathcal{DP}_1) & \text{Minimize} & \Delta t\,|u_0|^2 \\ & \text{subject to} & \dot{z}^\star = \frac{1}{2\pi i} \times \frac{k}{z} + u_0, \quad z(0) = z_0, \quad z(T) = z_f, \quad |u_0| \leq u_{max} \end{array}$$

This optimization problem is solved numerically with the Interior Points optimization algorithm [18], included in the `fmincon` solver o Matlab.

We want find $u_0 \in \mathbb{C}$ that moves the particle from $z_0 = -1-i$ and the final point $z_f = 2+2\,i$. We consider also a time of displacement $T = 10$ and a circulation of the vortex $k = 10$.

The control value obtained with the optimization solver `fmincon` is $u_0 = 0.189 - 0.179\,i$, that corresponds to a value of the objective function equal to 0.677979. In Figure 1 we can observe the trajectory of the particle obtained with this control.

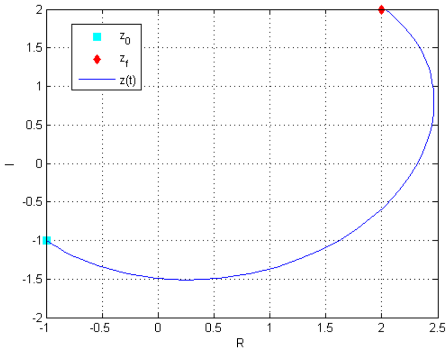

Figure 1: Optimal trajectory obtained with one control values.

The case $n = 2$ with u_0, $u_1 \in \mathbb{C}$. We now consider the case of two complex controls. We seek for a number $u_0,\ u_1 \in \mathbb{C}$ that drives the passive particle from z_0 to z_f in exactly T units of time and minimizes the energy. This problem is formulated as follows:

$$
\begin{array}{lll}
(\mathcal{DP}_2) & \text{Minimize} & \Delta t \sum_{k=0}^{1} |u_k|^2 \\
 & \text{subject to} & \dot{z}^\star = \frac{k}{2\pi i}\frac{1}{z} + u_0, \quad z(0) = z_0, \quad |u_0| \le u_{\max}, \quad t_0 \le t < t_1 \\
 & & \dot{z}^\star = \frac{k}{2\pi i}\frac{1}{z} + u_1, \quad z(t_1) = z_{t_1}, \ z(t_2) = z_f, \ |u_1| \le u_{\max}, \quad t_1 \le t < t_f
\end{array}
$$

This optimization problem is also solved numerically by means of the `fmincon` solver of Matlab. We want find $u_0 \in \mathbb{C}$ that moves the particle from $z_0 = -1 - i$ and the final point $z_f = 2 + 2\,i$. We consider also a time of displacement $T = 10$ and a circulation of the vortex $k = 10$.

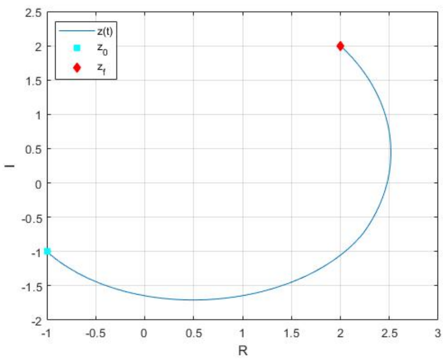

Figure 2: Optimal trajectory obtained with two control values.

The value obtained with the optimization solver `fmincon` is $u_0 = 0.0123 - 0.2491\,i$ and $u_1 = 0.1089 - 0.0151\,i$ and the value of the objective function is 0.371478. In Figure 2 we can observe the corresponding trajectory.

The case $n = 3$ with u_0, u_1, $u_2 \in \mathbb{C}$. We now consider the case of two complex controls. We seek for a number $u_0,\ u_1,\ u_2 \in \mathbb{C}$ that drives the passive particle from z_0 to z_f in exactly T

units of time and minimizes the energy. This problem is formulated as follows:

$$
\begin{array}{lll}
(\mathcal{DP}_3) & \text{Minimize} & \Delta t \sum_{k=0}^{2} |u_k|^2 \\
& \text{subject to} & \dot{z}^\star = \frac{k}{2\pi i}\frac{1}{z} + u_0, \quad z(0) = z_0, \quad |u_0| \leq u_{\max}, \quad t_0 \leq t < t_1 \\
& & \dot{z}^\star = \frac{k}{2\pi i}\frac{1}{z} + u_1, \quad z(t_1) = z_{t_1}, \quad |u_1| \leq u_{\max}, \quad t_1 \leq t < t_2 \\
& & \dot{z}^\star = \frac{k}{2\pi i}\frac{1}{z} + u_2, \quad z(t_2) = z_{t_2}, \; z(t_3) = z_f, \quad |u_2| \leq u_{\max}, \quad t_2 \leq t < t_f
\end{array}
$$

This optimization problem is also solved numerically by means of the `fmincon` solver of Matlab. We want find $u_0 \in \mathbb{C}$ that moves the particle from $z_0 = -1 - i$ and the final point $z_f = 2 + 2\,i$. We consider also a time of displacement $T = 10$ and a circulation of the vortex $k = 10$.

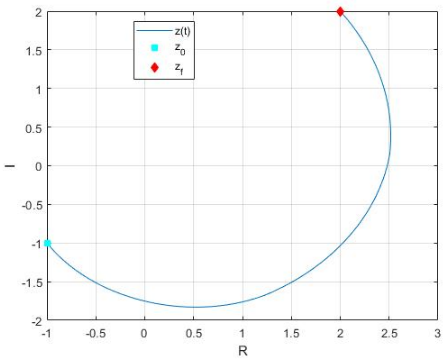

Figure 3: Optimal trajectory obtained with three control values.

The value obtained with the optimization solver `fmincon` is $u_0 = -0.1477 - 0.2297\,i$, $u_1 = 0.1171 - 0.0687\,i$ and $u_2 = 0.0951 + 0.0268\,i$ and the value of the objective function is 0.342648. In Figure 3 we can observe the corresponding trajectory.

The case $n = 4$ with $u_0, u_1, u_2, u_3 \in \mathbb{C}$. We now consider the case of two complex controls. We seek for a number $u_0,\, u_1,\, u_2,\, u_3 \in \mathbb{C}$ that drives the passive particle from z_0 to z_f in exactly T units of time and minimizes the energy. This problem is formulated as follows:

$$
\begin{array}{lll}
(\mathcal{DP}_4) & \text{Minimize} & \Delta t \sum_{k=0}^{3} |u_k|^2 \\
& \text{subject to} & \dot{z}^\star = \frac{k}{2\pi i}\frac{1}{z} + u_0, \quad z(0) = z_0, \quad |u_0| \leq u_{\max}, \quad t_0 \leq t < t_1 \\
& & \dot{z}^\star = \frac{k}{2\pi i}\frac{1}{z} + u_1, \quad z(t_1) = z_{t_1}, \quad |u_1| \leq u_{\max}, \quad t_1 \leq t < t_2 \\
& & \dot{z}^\star = \frac{k}{2\pi i}\frac{1}{z} + u_2, \quad z(t_2) = z_{t_2}, \quad |u_2| \leq u_{\max}, \quad t_2 \leq t < t_3 \\
& & \dot{z}^\star = \frac{k}{2\pi i}\frac{1}{z} + u_3, \quad z(t_3) = z_{t_3}, \; z(t_4) = z_f, \quad |u_3| \leq u_{\max}, \quad t_3 \leq t < t_f
\end{array}
$$

We want find $u_0 \in \mathbb{C}$ that moves the particle from $z_0 = -1 - i$ and the final point $z_f = 2 + 2\,i$. We consider also a time of displacement $T = 10$ and a circulation of the vortex $k = 10$. This optimization problem is also solved numerically by means of the `fmincon` solver of Matlab.

The value obtained with the optimization solver `fmincon` is $u_0 = -0.0806 - 0.2483\,i$, $u_1 = 0.1064 - 0.1301\,i$, $u_2 = 0.1103 - 0.0474\,i$ and $u_3 = 0.0769 + 0.0017\,i$ and the value of the objective function is 0.291819. In Figure 4 we can observe the corresponding trajectory.

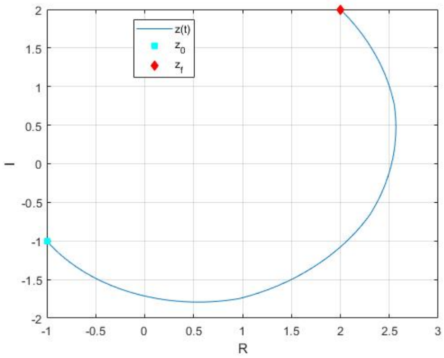

Figure 4: Optimal trajectory obtained with four control values.

4.2 Flow created by several vortices

In this Section we address the problem of a single passive particle ($P = 1$) moved by multiple vortices (N). We consider $N = 2$ in Subsection 4.2.1 , $N = 3$ in Subsection 4.2.2 and finally, $N = 4$ in Subsection 4.2.3. We solve these problems numerically by means of the Matlab solver `fmincon`.

4.2.1 Two vortices ($N = 2$) and one particle ($P = 1$).

In the two vortices and one particle problem, the vortices positions are given by [21]:

$$\begin{cases} z_1(t) = \frac{1}{k_1+k_2}\left[(k_1 z_1(0) + k_2 z_2(0)) + (z_1(0) - z_2(0))\, k_2 e^{i\,\Omega\, t}\right] \\ z_2(t) = \frac{1}{k_1+k_2}\left[(k_1 z_1(0) + k_2 z_2(0)) + (z_2(0) - z_1(0))\, k_1 e^{i\,\Omega\, t}\right] \end{cases} \tag{19}$$

where $\Omega = \frac{k1+k2}{2\pi D^2}$, $D = |z_2(0) - z_1(0)|$ and $z_1(0)$ and $z_2(0)$ are the initial position.

The passive particle position is given by the equation

$$\dot{z}^* = \frac{1}{2\pi i}\left(\frac{k_1}{z - z_1(t)} + \frac{k_2}{z - z_2(t)}\right) + u \tag{20}$$

with the given initial condition $z(0) = z_0$.

The optimization problem is similar to the one presented in Section 4.1.1 excepting the restriction given by the equation that describes the position of the particle that is replaced here by equations (19) and (20). As before we want find $u = [u_0, u_1, \ldots, u_{n-1}] \in \mathbb{C}^n$ that moves the particle from $z_0 = -1 - i$ and the final point $z_f = 2 + 2\,i$. We consider also a time of displacement $T = 10$ and the same circulation for the two vortices $k_1 = k_2 = 1$. The initial vortices positions are: $z_{10} = 0.5 + 0.5\,i$ and $z_{20} = 1.5 - 0.5\,i$. This problem is solved numerically with the Interior Points optimization algorithm [18], included in the `fmincon` solver.

Figure 5 presents two possible solution obtained with one ($n = 1$) and with a two control variable ($n = 2$). In both cases the targeted displacement corresponds to the solution given by the interior-point method. In the first case, Figure 5 a), the optimal solution is not find because $\min |z_f - z(T)| = 0.059844$. In the second case, Figure 5 b), the optimal solution is obtained and corresponds to $u_0 = 0.3185 - 0.1744\,i$ and $u_1 = 0.1808 - 0.3364\,i$ and the objective function is equal to 1.38819. These results show that there is an optimal control in the case of passive particle moved by a two vortices flow.

a) $n = 1$ **b)** $n = 2$

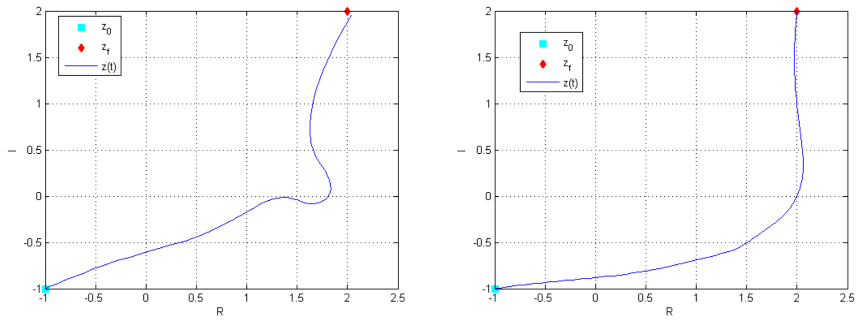

Figure 5: Two vortices problem: numerical solution. a) trajectory obtained with $n = 1$ control variable. b) trajectory obtained with $n = 2$ control variable.

4.2.2 Three vortices ($N = 3$) and one particle ($P = 1$).

In the problem with three vortices ($N = 3$) and one particle ($P = 1$), the vortices equations are

$$\begin{cases} \dot{z}_1^* = \frac{1}{2\pi i}\left(\frac{k_2}{z_1 - z_2} + \frac{k_3}{z_1 - z_3}\right) \\ \dot{z}_2^* = \frac{1}{2\pi i}\left(\frac{k_1}{z_2 - z_1} + \frac{k_3}{z_2 - z_3}\right) \\ \dot{z}_3^* = \frac{1}{2\pi i}\left(\frac{k_1}{z_3 - z_1} + \frac{k_2}{z_3 - z_2}\right) \end{cases} \tag{21}$$

with the given initial conditions $z_1(0) = z_{10}$, $z_2(0) = z_{20}$, and $z_3(0) = z_{30}$.

The passive particle equation is

$$\dot{z}^* = \frac{1}{2\pi i}\left(\frac{k_1}{z - z_1} + \frac{k_2}{z - z_2} + \frac{k_3}{z - z_3}\right) + u \tag{22}$$

with the given initial condition $z(0) = z_0$.

The optimization problem is similar to the one presented in previous Section 4.1.1. The restriction due to the position of the particle is now given by the initial value problem that includes equations (21) and (22) and respective initial conditions. As before we want find $u = [u_0, u_1, \ldots, u_{n-1}] \in \mathbb{C}^n$ that moves the particle from $z_0 = -1 - i$ to the final point $z_f = 2 + 2\,i$ in exactly $T = 10$ unities of time. We consider also the same circulation for the three vortices $k_1 = k_2 = k_3 = 1$. The initial vortices positions are: $z_{10} = 0.5 + 0.5\,i$, $z_{20} = 1.5 - 0.5\,i$ and $z_{30} = 1 + i$. This problem is solved numerically with the Active-Set optimization algorithm [19, 22], included in the `fmincon` solver.

Figure 6 presents the solution obtained with $n = 1, 2, 3$ and 4. In all the cases the optimal solution is obtained. The objective function values equal to a) 137.526, b) 89.329, c) 83.497 and d) 84.245. These results show us that there is optimal control for this problem and that the value of the objective function tends to decrease as the number of control variables increase.

4.2.3 Four vortices ($N = 4$) and one particle ($P = 1$).

We address now the case of four vortices ($N = 4$) and one particle ($P = 1$). This is a very interesting case because it is considered as a chaotic advection in point vortex models and two-dimensional turbulence [15].

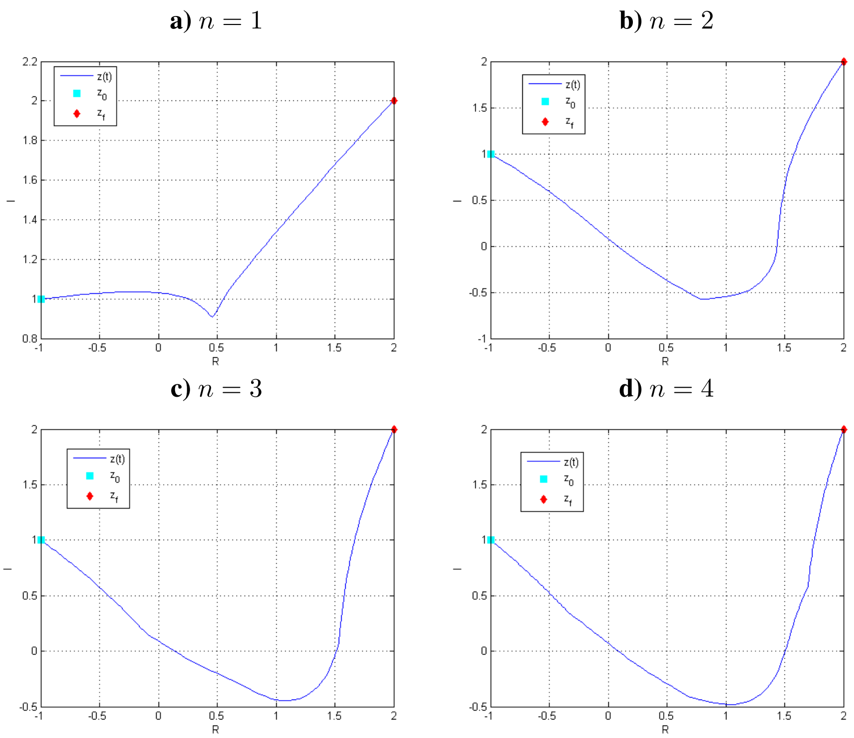

Figure 6: Three vortices problem: numerical solution. Trajectory obtained with $n = 1$ a), $n = 2$ b), $n = 3$ c) and $n = 4$ d) control variable.

The vortices equations are

$$\begin{cases} \dot{z}_1^* = \frac{1}{2\pi i}\left(\frac{k_2}{z_1 - z_2} + \frac{k_3}{z_1 - z_3} + \frac{k_4}{z_1 - z_4}\right) \\ \dot{z}_2^* = \frac{1}{2\pi i}\left(\frac{k_1}{z_2 - z_1} + \frac{k_3}{z_2 - z_3} + \frac{k_4}{z_2 - z_4}\right) \\ \dot{z}_3^* = \frac{1}{2\pi i}\left(\frac{k_1}{z_3 - z_1} + \frac{k_2}{z_3 - z_2} + \frac{k_4}{z_3 - z_4}\right) \\ \dot{z}_4^* = \frac{1}{2\pi i}\left(\frac{k_1}{z_4 - z_1} + \frac{k_2}{z_4 - z_2} + \frac{k_3}{z_4 - z_3}\right) \end{cases} \tag{23}$$

with the given initial positions $z_1(0) = z_{10}$, $z_2(0) = z_{20}$, $z_3(0) = z_{30}$ and $z_4(0) = z_{40}$. The passive particle equation is

$$\dot{z}^* = \frac{1}{2\pi i}\left(\frac{k_1}{z - z_1} + \frac{k_2}{z - z_2} + \frac{k_3}{z - z_3} + \frac{k_4}{z - z_4}\right) + u \tag{24}$$

with the given initial condition $z(0) = z_0$. As in the two previous problems, equations (23) and (24) and respective initial conditions gives rise to the initial value problem that it is included in the optimization problem as restriction.

We want find $u = [u_0, u_1, \ldots, u_{n-1}] \in \mathbb{C}^n$ that moves the particle from $z_0 = -1 - i$ to the final point $z_f = 2 + 2\,i$ in exactly $T = 10$ unities of time. We consider as before the same values for the circulation for the four vortices $k_1 = k_2 = k_3 = k_4 = 1$. The initial vortices positions are: $z_{10} = 0.5 + 0.5\,i$, $z_{20} = 1.5 - 0.5\,i$, $z_{30} = 1 + i$ and $z_{40} = -1 - 2\,i$. This problem is solved numerically with the Interior Points optimization algorithm [18], included in the `fmincon` solver.

Figure 7 presents the solution obtained in the case of four vortices with $n = 1, 2, 3$ and 4 control variables. In all the cases the optimal solution is obtained because $\min|z_f - z(T)|$ is

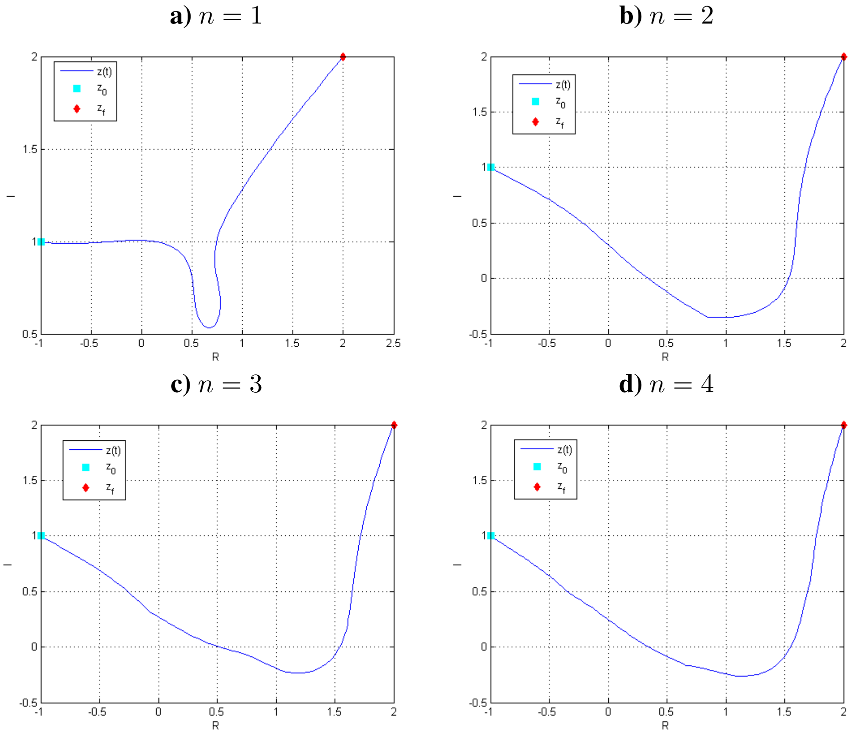

Figure 7: Four vortices problem: numerical solution. Trajectory obtained with $n = 1$ a), $n = 2$ b), $n = 3$ c) and $n = 4$ d) control variable.

close to zero. The objective function values equal to a) 112.278, b) 66.014, c) 61.174 and d) 61.937. These results show us that there is an optimal control for this problem and that the value of the objective function tends to decrease until the number of control variables increase to 3.

5 CONCLUSIONS

The singular solutions of the two-dimensional incompressible Euler equations are known as point vortices. Point vortices are used to describe the dynamic of vortex-dominated flows. Because they are based on a low dimensional description of the flow features and, consequentially, enables the solution of fluid dynamic problems with low computational costs.

By definition, a passive tracer is a point vortex with zero circulation. In our case, we consider the advection of one passive tracer by point vortices in the unbounded plane and we have presented the formulation of corresponding control problems. These control problems results from the necessity of displacing the particle between two point in a fixed interval of time.

We use a control strategy, based on a direct approach, that is unusual in these kind of problems. This approach enables to work with vortex dynamics resulting from the interaction of several point vortices. In our case we consider dynamics induced by $N = 1$, $N = 2$, $N = 3$ and $N = 4$ vortices.

We have discretize the main time interval in a certain number of subintervals, wherein the control variable is constant, and we solve numerically the resulting non-linear programming. The results show the existence of optimal controls for the cases of $N = 1$, $N = 2$, $N = 3$ and $N = 4$ vortices. The number of suitable control variables varies from problem to problem. However, there is a tendency for this number to increase with the number of point vortices.

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