Computing ODE Symmetries from a Variational Point of View

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Introduction

Sophus Lie was the first to introduce the use of symmetries into the study of differential equations; Emmy Noether the first to recognize the important role of symmetries in the calculus of variations. Nowadays, all the computer algebra systems which deal with differential equations provide several tools to help the user with the analysis of Lie symmetries. Recently, the authors developed a computer algebra package for the automatic computation of Noether variational symmetries in the calculus of variations [2] and optimal control [3], now available as part of the Maple Application Center at [http://www.maplesoft.com/applications/app_center_view.aspx?AID=1983].

The omnipresent tools for Lie symmetries provide a great help for the search of solutions of ODEs, their classification, order reduction, proof of integrability, or in the construction of first integrals. From the mathematical point of view, an ODE symmetry is described by a group of transformations that keep the ordinary differential equation invariant. Depending on the type of transformations one is considering, different symmetries are obtained. An important class of symmetries is obtained considering a one-parameter family of transformations which form a local Lie group. Those transformations are often represented by a set of functions known as the infinitesimal generators. From the practical point of view, the determination of the infinitesimal generators that define a symmetry for a given ODE is, in general, a complex task [1,4]. To address the problem, we follow a different approach. We propose a new method for computing symmetries of ODEs by using a Noetherian perspective. Making use of our previous algorithm [3], that has shown up good results for the computation of Noether variational symmetries of problems of the calculus of variations and optimal control, we look to an ODE as being the control system of an optimal control problem. Then, we obtain symmetries for the ODE by computing the abnormal variational symmetries of the associated optimal control problem.

About the algorithm

We use the CAS Maple to define a new computational procedure that determines, in an automatic way, symmetries of ODEs. The automatic calculation of symmetries is a subject much studied under the theory of differential equations, with many results and applications in many different areas. Our main novelty is the presentation of a new algorithm, alternative to existing ones, which looks to symmetries of ODEs as particular cases of Noether-variational symmetries. Our algorithm involves the resolution of a first order, homogeneous, and linear PDE, which is the abnormal case of the necessary and sufficient condition of invariance for problems of optimal control studied in connection with Noether's theorem [3], [5]. Interesting points of the proposed method are: (i) it is based on a new approach to the subject - in particular, it is different from all the nine alternative algorithms available in Maple; (ii) allows us to get dynamic symmetries for ODEs of any order; (iii) allows to determine symmetries for systems of ODEs, when the analog *simgen* Maple command of the *DEtools* package can only obtain solutions for a single ODE.

Roughly speaking, our algorithm consists to look to an ODE as a control system. In order to use the formalism of optimal control and the notion of variational symmetry [3], one thing is missing: the existence of an integral functional to be minimized, and whose Lagrangian L enters in the definition of the Hamiltonian, thus being necessary for computing variational symmetries. However, if we restrict ourselves to the abnormal case, where the cost functional has no role, then the Hamiltonian H does not depend on the Lagrangian L and we can look to our system as an optimal control problem. In other words, if we only consider the abnormal case, the control system (the ODE) is everything one needs in order to find symmetries. Concluding: to ODE symmetries we associate a smaller group of infinitesimal transformations than that used in optimal control - we are only interested in transformations of the independent variable and transformations of the dependent variables (an ODE is a particular case of a control system where the velocity vector does not depend on the control variables). The complete Maple definitions of our new procedure *odeSymm* can be obtained from [http://www.mat.ua.pt/delfim/maple.htm].

Illustrative examples

In order to show the functionality and the usefulness of our new Maple procedure *odeSymm*, we consider here three concrete problems.

Example 1. We begin with a first order ODE:

```
> ode:= t*diff(y(t),t)-y(t)*(t*ln(t^2/y(t))+2)=0:
```

To obtain symmetries of the equation we use our Maple procedure *odeSymm:*

```
> gerad:= odeSymm(ode, y(t), split, hint=nohint):
```

One can test the validity of the obtained symmetries with the *symtest* command of the *DEtools* Maple package:

```
> map(DEtools[symtest], [gerad], ode, y(t));
                                    [0,0]
```

The *symtest* confim that the infinitesimal generators leave the given ODE invariant, i.e., the generators obtained by our method give indeed a symmetry. It is interesting to remark that, without the knowledge of the computed symmetries, the ODE Maple solver *dsolve* is not able to integrate the ODE:

```
> dsolve(ode, y(t), class);
```

However, when one gives to the Maple solver the infinitesimal generators found by our method, the ODE is correctly solved:

```
> dsolve(ode, y(t), HINT=[gerad]);
```

$$y(t) = t^2 * e^{(C_1 - 1) * e^{-t}}$$

Example 2. [Damped Harmonic Oscillator] We consider a harmonic oscillator with restoring force *-kx*, emersed in a liquid in such a way that the motion of the mass *m* is damped by a force proportional to its velocity. Using Newton's second law one obtains, as the equation of motion, the following second order differential equation:

```
> EL:= m*diff(x(t),t,t)+a*diff(x(t),t)+k*x(t)=0:
```

The symmetries for this equation are easily obtained with our Maple procedure *odeSymm:*

```
> gerad:= odeSymm(EL, x(t), split):
```

One can confirm that these infinitesimals represent valid symmetries for the differential equation:

```
> map(DEtools[symtest], [gerad], EL, x(t));
                                          [0, 0, 0, 0, 0]
```

For this example our *odeSymm* procedure gives a dynamical symmetry: the derivative of the dependent variable is present in one of the obtained infinitesimal generators.

Example 3. [Kepler's problem] In this case the Lagrangian depends on two dependent variables *q1* and *q2*:

```
> L:= m/2*(v[1]^2+v[2]^2)+K/sqrt(q[1]^2+q[2]^2):
```

We compute symmetries for the corresponding Euler-Lagrange differential equation:

```
> EL:= EulerLagrange(L, t, [q[1],q[2]], [v[1],v[2]]):
```

Our *odeSymm* procedure is able to determine the symmetries through the command:

```
> odeSymm(EL, [q[1](t),q[2](t)], split):
```

This example can not be handled by the algorithms available in Maple. Indeed, the Maple command *symgen* that looks for a symmetry generator for a given ODE is not able to deal with more than one dependent variable. On contrary, our *odeSymm* procedure is able to determine symmetries for systems of differential equations as well.

References

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