Growth of number of periodic orbits of one family of skew product maps

Salete Esteves

School of Public Management, Communication and Tourism,
Polytechnic lnstitute of Bragança
e-mail: saleteesteves@ipb.pt

Abstract

In this article we introduce a one-parameter family of skew product $(G_t)_{t\in[-\epsilon,\epsilon]}$ maps exhibiting a heterodimensional cycle such that the number of isolated periodic orbits inside it has not super-exponential growth. The dynamics in the central direction of the maps G_t is described by a one-parameter family of system of iterated functions.

Keywords:. Skew-product maps, Artin-Mazur maps, heterodimensional cycle, homoclinic class, index of a saddle.

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1 Introduction

Let M be a compact manifold of dimension greater or equal to 2 and, for $r \in \mathbb{N}$, $C^r(M,M)$ be the space of C^r mappings of M into itself, endowed by the C^r topology. Given $f \in C^r(M,M)$, for each $m \in \mathbb{N}$, we consider the set of isolated fixed points of f^m, i.e. the set

$$\mathbb{P}_m(f) := \{x \in M : x \text{ is an isolated fixed point of } f^m\}.$$

We say that f is *Artin-Mazur*, A-M for short, if $\#\mathbb{P}_m(f)$ grows at most exponentially fast, i.e. there is a constant $K > 0$ such that

$$\#\mathbb{P}_m(f) \leq exp(Km), \quad \text{for all } m \in \mathbb{N}.$$

In [AM] Artin and Mazur proved that the set of these maps are dense in the space of $C^r(M,M)$.

Let $\mathrm{Diff}^r(M)$ be the space of C^r diffeomorphisms of a smooth compact manifold M. In [K1], Kaloshin proved that the set of A-M diffeomorphisms having only hyperbolic periodic orbits is dense in $\mathrm{Diff}^r(M)$.

We recall that a *Newhouse Domain* $\mathcal{N} \subseteq \mathrm{Diff}^r(M)$ is an open set where diffeomorphisms exhibiting homoclinic tangencies are dense (see [N]). In [K2], Kaloshin proved the superexponential growth of periodic points for diffeomorphisms in Newhouse domains, more precisely he proved that, for an arbitrary sequence of positive integers $a = (a_n)_{n\in\mathbb{N}}$, there exists a residual set $\mathcal{R}_a \subset \mathcal{N}$, with the property that, for $f \in \mathcal{R}_a$,

$$\limsup_{n\to\infty} \frac{\#\mathbb{P}_n(f)}{a_n} = \infty.$$

Let us recall that the *index* of an hyperbolic periodic point is the dimension of the unstable manifold, and that the *homoclinic class* of a hyperbolic saddle P of a diffeomorphism f, denoted by $H(P,f)$, is the closure of the transverse intersections of the invariant manifolds (stable and unstable ones) of the orbit of P. It is clear that homoclinic classes having periodic points of different indices can not be hyperbolic.

In [BDF] Bonatti, Díaz, and Fisher proved the super-exponential growth of periodic points in the setting of non-hyperbolic homoclinic classes. More precisely, in [BDF] they prove that there is a C^1-residual subset $S(M)$ of $\mathrm{Diff}^1(M)$ such that, for every $f \in S(M)$, any homoclinic class of f containing saddles of different indices has superexponential growth of the number of periodic points. Kaloshin and Kozlovski in [KK] constructed an example of C^r unimodal map on the unit interval whose number of periodic points grows faster than any given sequence along a subsequence $n_k = 3^k$. In this paper, following the model of [DER], we present systems G_t with a heterodimensional cycle at $t = 0$ and we prove that the growth of the number of periodic orbits for this systems is at most exponential (see Theorem 1.1 for the precise statement).

Let f be a diffeomorphism, defined on a closed manifold of dimension equal to $n \in \mathbb{N}$, with two hyperbolic periodic points P and Q with indices p and $n-p+1$, respectively. We say that f has a *heterodimensional cycle* associated to P and Q if the stable manifold $W^s(P, f)$ intersects the unstable manifold $W^u(Q, f)$ of Q, and the same for $W^u(P, f)$ and $W^s(Q, f)$. The heterodimensional cycles were first considered by Newhouse and Palis in [NP] and were studied systematically in the series of papers [D1, D2, DR1].

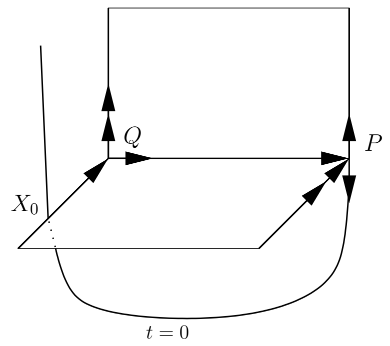

Figure 1: The heterodimensional cycle

The heuristic principle in [D1, D2, DR1] is that the dynamics in the neighborhood of the cycle, after of the unfolding of a heterodimensional cycle, is mainly determined by the study of one-dimensional one-parameter families of *iterated functions systems* which describe the central dynamics.

In fact, the quotient dynamics along the strong stable and strong unstable directions consists of an iterated function system with two generators: the restriction of f to the central direction and a translation. Roughly speaking, the construction of these families is a skew product over a shift with two symbols and that several sequences of 0's and 1's are forbidden.

Consider the space of bi-infinite sequence of two-symbols $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$ endowed with the standard metric and the bernoulli shift function

$$\begin{array}{rccc} \sigma : & \Sigma_2 & \to & \Sigma_2 \\ & \xi = (\xi_i)_{i \in \mathbb{Z}} & \mapsto & \sigma(\xi) = (\xi_{i+1})_{i \in \mathbb{Z}}. \end{array}$$

Here, we consider a one-parameter family $(G_t)_{t \geq 0}$ of skew product maps

$$G_t : \Sigma_2 \times [-1, 1] \to \Sigma_2 \times \mathbb{R}, \; G_t(\xi, x) = (\sigma(\xi), g_{\xi_0, t}(x)) \tag{1.1}$$

where the fiber maps $g_{0,t}$ and $g_{1,t}$ satisfy the following conditions:

(h1) The *central map* g_0 of class C^2 is independent of t ($g_0 = g_{0,t}, \; \forall t$), it has two fixed points, a repeller 0 and an attractor 1/2, and, for some $\epsilon > 0$, $g_0'(x) > 0$ and $g_0''(x) < 0$ for all $x \in [-\epsilon, 1/2 + \epsilon]$;

(h2) The *cycle maps* $g_{1,t}$ are defined by

$$g_{1,t}(x) = (x - 1/2) + t.$$

Note that $g_{1,0}$ maps the attractor point 1/2 into the repeller point 0.

For every $t \geq 0$, the points $Q = (0^{\mathbb{Z}}, 0)$ and $P = (0^{\mathbb{Z}}, 1/2)$ are fixed points of G_t and

$$\begin{array}{c} \{0^{\mathbb{Z}}\} \times (0, 1/2) \subset W^u(Q, G_t) \cap W^s(P, G_t) \\ (0^{-\mathbb{N}}.10^{\mathbb{N}}, 1/2) \in W^s(Q, G_0) \cap W^u(P, G_0) \end{array} \tag{1.2}$$

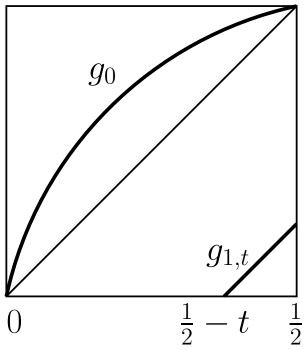

Figure 2: The maps $g_0|_{[0,1/2]}$ and $g_{1,t}|_{[1/2-t,1/2]}$

Therefore the stable and unstable invariant sets intersect cyclically for G_0. As the points P and Q have different central behaviour (contracting and repelling, respectively), one can think that G_0 has two hyperbolic points with different indices and therefore G_0 has a heterodimensional cycle associated to these two points.

A *neighborhood of the cycle* of G_0 associated to P and Q is an open set $\mathcal{V}$ that contains the points P and Q, the set $\{0^{\mathbb{Z}}\} \times (0, 1/2) \subseteq W^s(P, G_0) \cap W^u(Q, G_0)$ and the orbit of $(0^{-\mathbb{N}}.10^{\mathbb{N}}, 1/2) \in W^s(Q, G_0) \cap W^u(P, G_0)$. In this paper, we will build a convenient neighborhood of the cycle of G_0, $\mathcal{V}$, and focus our attention in the *maximal invariant set of* G_t in $\mathcal{V}$, that is, in the set

$$\Lambda_t(\mathcal{V}) := \bigcap_{n\in\mathbb{Z}} G_t^n(\mathcal{V}), \tag{1.3}$$

for $t \geq 0$ small enough.

Although the study of the skew product maps appear as a modulation of heterodimensional cycles, their study is important by itself. The role of the skew products is similar to the one of the shift for the study of the horseshoe. As in heterodimensional cycles, the fiber dynamics are given by a system $\mathfrak{F}_t$ of iterated functions. In fact, this is an important tool of this work.

The main result of this paper can now be stated as follows:

Theorem 1.1. *Let $(G_t)_{t\geq 0}$ be the family of skew product maps* (1.1) *such that* $(h1)$ *and* $(h2)$ *hold. Then there is a neighborhood of the cycle of G_0, $\mathcal{V}$, such that, for any $t > 0$ small enough there exists $n_0 = n_0(t) \in \mathbb{N}$ satisfying*

$$\#\mathbb{P}_m(G_t|_{\Lambda_t(\mathcal{V})}) \leq m \cdot 2^{m-(n_0+1)} + 2, \quad \textit{for all } m \geq n_0 + 2, \tag{1.4}$$

where $\Lambda_t(\mathcal{V})$ is defined by (1.3)*.*

As it is illustrated in section 4 with the example 4.4, it is possible to choose a central map g_0 such that $H_{\mathcal{V}}(P, G_t) = H_{\mathcal{V}}(Q, G_t)$.

This paper is organized as follows. In section 2 we present some terminilogy about skew product maps. In section 3 we introduce the iterated function system (IFS) associated to the skew product and we study the grow of periodic points in the IFS. The study of the dynamics in a neighborhood of the cycle using the IFS and the prof of the main result is done in section 4.

2 Notations and definitions

In this section we introduce and make the natural adaptations of some definitions introduced for heterodimensional cycles. For the sake of completeness see [DER].

Consider a skew product map

$$G : \Sigma_2 \times [-1, 1] \to \Sigma_2 \times \mathbb{R}, \; G(\xi, x) = (\sigma(\xi), g_{\xi_0}(x)),$$

where $g_0, g_1 : [-1,1] \to \mathbb{R}$ are two differentiable functions. As the range of g_i, $i = 0,1$, can not be $[-1,1]$, it follows that some of the compositions $g_i \circ g_j$ are not defined.

For $m \in \mathbb{N}_0$, we say that $X = (\xi, x) \in \Sigma_2 \times [-1,1]$ is a *periodic point* of G of period $m+1$ if $\sigma^{m+1}(\xi) = \xi$ and $(g_{\xi_m} \circ \cdots \circ g_{\xi_0})(x) = x$. Since g_i, $i = 0,1$, are differentiable, the map G is differentiable on the second variable (the *fiber direction*) and a periodic point $X = (\xi, x)$ of G of period $m+1$ is called *hyperbolic* if x is a hyperbolic fixed point of $g_{\xi_m} \circ \cdots \circ g_{\xi_0}$, that is,

$$(g_{\xi_m} \circ \cdots \circ g_{\xi_0})'(x) \neq \pm 1.$$

A hyperbolic periodic point X is of *contracting type* if this derivative has modulus less than one, otherwise this point is of *expanding type.*

Now we fix some notation. For $m \in \mathbb{N}$ and $\xi_0, \ldots, \xi_m \in \{0,1\}$, the associated *cylinder map* is defined by

$$g_{[\xi_0 \cdots \xi_m]} := g_{\xi_m} \circ \cdots \circ g_{\xi_0}.$$

Given a hyperbolic fixed point a of $g_{[\xi_0 \cdots \xi_m]}$, consider its local stable manifold, $W^s_{\text{loc}}(a, g_{[\xi_0 \cdots \xi_m]})$ and its local unstable manifold, $W^u_{\text{loc}}(a, g_{[\xi_0 \cdots \xi_m]})$. Observe that if a is contracting (respectively expanding), we have $W^u_{\text{loc}}(a, g_{[\xi_0 \cdots \xi_m]}) = \{a\}$ (respectively $W^s_{\text{loc}}(a, g_{[\xi_0 \cdots \xi_m]}) = \{a\}$).

For $n \in \mathbb{N}$ and $\alpha \in \{0,1\}$, we denote by α^n the finite sequence $\underbrace{\alpha \cdots \alpha}_{n \text{ times}}$ and, for $k,\ l,\ r,\ m \in \mathbb{N}$ and $\delta_{-r}, \ldots, \delta_{-1}, \eta_{-l}, \ldots, \eta_k, \alpha_1, \ldots, \alpha_m \in \{0,1\}$, we define the set

$$[\eta_{-l} \cdots \eta_{-1}.\eta_0 \cdots \eta_k] := \{(\xi_i)_{i \in \mathbb{Z}} \in \Sigma_2 : \xi_i = \eta_i \text{ if } i \in \{-l, \ldots, -1, 0, 1, \ldots, k\}\}$$

and we denote by

$$\big((\delta_{-r} \cdots \delta_{-1})^{-\mathbb{N}} \eta_{-l} \cdots \eta_{-1}.\eta_0 \cdots \eta_k (\alpha_1 \cdots \alpha_m)^{\mathbb{N}}\big)$$

the sequence $\xi = (\xi_i)_{i \in \mathbb{Z}}$ defined by

- $\xi_i = \eta_i$, for $i \in \{-l, \ldots, k\}$;
- $\xi_{k+(s-1)m+i} = \alpha_i$, for every $i \in \{1, \ldots, m\}$ and $s \in \mathbb{N}$,
- $\xi_{-l-(s-1)r-i} = \delta_{-i}$ for every $i \in \{1, \ldots, r\}$ and $s \in \mathbb{N}$.

We denote by $\xi = (\xi_0 \cdots \xi_m)^{\mathbb{Z}}$ the periodic sequence of period $m+1$, $\xi = (\xi_i)_{i \in \mathbb{Z}} \in \Sigma_2$, defined by $\xi = \big((\xi_0 \cdots \xi_m)^{-\mathbb{N}}.(\xi_0 \cdots \xi_m)^{\mathbb{N}}\big)$. Note that if $\xi \in \Sigma_2$ is a periodic sequence of period $m+1$, then ξ is a fixed point of σ^{m+1}, that is, ξ is a periodic point of period $m+1$ of σ. Let $A = \big((\xi_0 \cdots \xi_m)^{\mathbb{Z}}, a\big)$ be a periodic point of G of period $m+1$. We define the stable and unstable sets as

$$W^s(A,G) = \left\{ \Big(\big(\cdots .\eta_0 \cdots \eta_k (\xi_0 \cdots \xi_m)^{\mathbb{N}}\big) ; x \Big) : g_{[\eta_o \cdots \eta_k]}(x) \in W^s_{\text{loc}}\left(a, g_{[\xi_0 \cdots \xi_m]}\right) \right\};$$
$$W^u(A,G) = \left\{ \Big(\big((\xi_0 \cdots \xi_m)^{-\mathbb{N}} \eta_{-k} \cdots \eta_{-1}. \cdots \big) ; x \Big) : g^{-1}_{[\eta_{-1} \cdots \eta_{-k}]}(x) \in W^u_{\text{loc}}\left(a, g_{[\xi_0 \cdots \xi_m]}\right) \right\}.$$

Definition 2.1 (heterodimensional cycle or cycle)**.** *We say that a pair of periodic points, A and B, of G (of different type) has a* heterodimensional cycle *if*

$$W^u(A,G) \cap W^s(B,G) \neq \emptyset \text{ and } W^s(A,G) \cap W^u(B,G) \neq \emptyset.$$

At last, we introduce a further definition.

Definition 2.2. *The* homoclinic class *of a hyperbolic point A of G, denoted by $H(A,G)$, is the closure of the intersections of the invariant sets $W^s(A,G)$ and $W^u(A,G)$ of A. Given a neighborhood U of the orbit of a periodic point A,* the relative homoclinic class of A to U, *denoted by $H_U(A,G)$, is the subset of $H(A,G)$ of points whose orbit is contained in U.*

3 One-dimensensional dynamics. Iterated function systems

In this section we introduce the iterated function system (IFS) associated to the skew product (1.1) and study the grow of periodic points in the IFS.

3.1 Returns and iterated in the function systems

For small $t > 0$, define $d_t \in (t, g_0(t)]$ and $n_0 = n_0(t) \in \mathbb{N}$ by

$$g_{1,t} \circ g_0^{n_0}(d_t) = 0, \text{ that is, } g_0^{n_0}(d_t) = 1/2 - t,$$

and consider the *fundamental domain* of g_0 given by

$$D_t = (g_0^{-1}(d_t), d_t]. \tag{3.1}$$

Observe that D_t varies continuously with t and $d_t \to 0$ as $t \to 0$. By construction, $g_0^{n_0+1}(D_t) = (1/2 - t, g_0(1/2 - t)]$ and we say that $n_0 + 1$ is the *transition time from* 0 *to* 1/2.

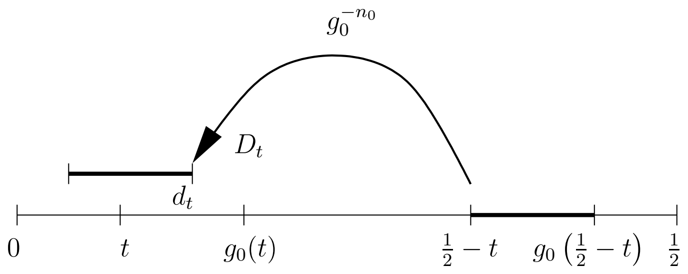

Figure 3: The fundamental domain D_t

For $x \in D_t$ and $s \in \mathbb{N}_0$ we have

$$0 < g_{1,t} \circ g_0^{n_0+1+s}(x) < g_{1,t}(1/2) = t < d_t. \tag{3.2}$$

Thus, for each $s \in \mathbb{N}_0$, there is exactly one $u_s(x) \in \mathbb{N}_0$ such that

$$g_0^{u_s(x)} \circ g_{1,t} \circ g_0^{n_0+1+s}(x) \in D_t.$$

This leads us to define, for each $(u, s) \in \mathbb{N}_0 \times \mathbb{N}_0$, the *return map* by

$$\Gamma_t^{(u,s)} : D_t^{(u,s)} \to D_t, \qquad \Gamma_t^{(u,s)}(x) \;\; := \;\; g_0^u \circ g_{1,t} \circ g_0^{n_0+1+s}(x) \tag{3.3}$$

where $D_t^{(u,s)}$ is the maximal subinterval of D_t where $\Gamma_t^{(u,s)}$ is defined, that is,

$$D_t^{(u,s)} = D_t \cap (g_0^u \circ g_{1,t} \circ g_0^{n_0+1+s})^{-1}(D_t).$$

Note that, it is possible there may be a pair (u, s) for which $D_t^{(u,s)}$ is a empty set. By definition, for each $s \in \mathbb{N}_0$, one has

$$D_t = \bigcup_{u \geq 0} D_t^{(u,s)} \text{ and } D_t^{(u,s)} \cap D_t^{(u^\star,s)} = \emptyset \text{ if } u \neq u^\star.$$

Given $l \in \mathbb{N}$ and a *chain* $\mathfrak{b} = \mathfrak{b}_l = (u_1, s_1) \cdots (u_l, s_l)$, with $(u_i, s_i) \in \mathbb{N}_0 \times \mathbb{N}_0$ and $i = 1, \ldots, l$, we define the *l-return map* associated to $\mathfrak{b}$ as the map

$$\Gamma_t^{\mathfrak{b}} : D_t^{\mathfrak{b}} \to D_t, \quad \Gamma_t^{\mathfrak{b}}(x) := \Gamma_t^{(u_l,s_l)} \circ \cdots \circ \Gamma_t^{(u_1,s_1)}(x), \tag{3.4}$$

where $D_t^{\mathfrak{b}} \subseteq D_t$ is the maximal domain of definition of $\Gamma_t^{\mathfrak{b}}$ (this set may be empty). If $D_t^{\mathfrak{b}} \neq \emptyset$, then we say that the chain $\mathfrak{b}$ is *admissible*.

As an immediate consequence of the definitions of g_0 and $g_{1,t}$ we get the following result:

Lemma 3.1. *For every admissible chain $\mathfrak{b}$, the map $\Gamma_t^{\mathfrak{b}} : D_t^{\mathfrak{b}} \to D_t$ has at most two fixed points.*

Proof. First note that the maps $\Gamma_t^{\mathfrak{b}}$ are compositions of g_0 and $g_{1,t}$ which are increasing function on $[0,1/2]$ and on $[1/2-t,1/2]$ respectively, therefore the maps $\Gamma_t^{\mathfrak{b}}$ are also increasing. Moreover, as g_0' is a strictly decreasing map on $[0,1/2]$ and $g_{1,t}'(x)=1$, the map $\left(\Gamma_t^{\mathfrak{b}}\right)'$ is also strictly decreasing and consequently it has at most two fixed points on $D_t^{\mathfrak{b}}$. □

Remark 3.2. *From the proof of Lemma 3.1 we know that, if $\mathfrak{b}$ is an admissible chain, then $\Gamma_t^{\mathfrak{b}}$ is an increasing function and $\left(\Gamma_t^{\mathfrak{b}}\right)'$ is a decreasing function.*

3.2 The system of iterated function $\mathfrak{F}_t$ and its periodic points

Given an admissible chain $\mathfrak{b} = (u_1, s_1)\cdots(u_l, s_l)$, $l \in \mathbb{N}$, we associate the finite sequence

$$\theta(\mathfrak{b}) := 0^{n_0+1+s_1}10^{u_1}0^{n_0+1+s_2}10^{u_2}\cdots 0^{n_0+1+s_l}10^{u_l}, \tag{3.5}$$

and define the length of $\theta(\mathfrak{b})$ as follows

$$|\theta(\mathfrak{b})| := \sum_{i=1}^{l}(s_i + n_0 + 1 + 1 + u_i) = l(n_0+2) + \sum_{i=1}^{l}(s_i + u_i). \tag{3.6}$$

The one-parameter family of *iterated function systems* (IFS), $\mathfrak{F}_t$, is defined by

$$\mathfrak{F}_t := \{\Gamma_t^{\mathfrak{b}} : \mathfrak{b} \text{ is an admissible chain}\}, \tag{3.7}$$

and we say that $x \in D_t$ is a *periodic point* of the system $\mathfrak{F}_t$ if there is an admissible chain $\mathfrak{b} = (u_1, s_1)\cdots(u_l, s_l)$ such that $\Gamma_t^{\mathfrak{b}}(x) = x$. Naturally, there may be a periodic point of $\mathfrak{F}_t$, $x \in D_t$, such that

$$\Gamma_t^{\mathfrak{b}}(x) = \Gamma_t^{\mathfrak{b}'}(x)$$

for two different admissible chains $\mathfrak{b}$ and $\mathfrak{b}'$. Thus we denote the set

$$\mathbb{P}(\mathfrak{F}_t) := \{(\mathfrak{b}, x) : \Gamma_t^{\mathfrak{b}}(x) = x, \text{ for } \Gamma_t^{\mathfrak{b}} \in \mathfrak{F}_t\}$$

and, for each $m \in \mathbb{N}$, we define the set

$$\mathbb{P}_m(\mathfrak{F}_t) := \{(\mathfrak{b}, x) : \Gamma_t^{\mathfrak{b}}(x) = x, \text{ for some } \Gamma_t^{\mathfrak{b}} \in \mathfrak{F}_t \text{ and } |\theta(\mathfrak{b})| = m\}. \tag{3.8}$$

Observe that $\mathbb{P}_m(\mathfrak{F}_t) = \emptyset$ if $m < n_0 + 2$ (see (3.6)).

In section 4, we will prove that $\#\mathbb{P}_m(G_t|_{\Lambda_t(\mathcal{V})}) \le m \cdot \#\mathbb{P}_m(\mathfrak{F}_t) + 2$ for a convenient neighborhood of the cycle of G_0, $\mathcal{V}$ (independent of t), $t > 0$ small enough, and $m \ge n_0 + 2$ (Proposition 4.2 below). Therefore the following theorem implies the inequality (1.4) in Theorem 1.1.

Theorem 3.3. *Let $t > 0$ small. If $m \ge n_0 + 2$, then*

$$\#\mathbb{P}_m(\mathfrak{F}_t) \le 2^{m-(n_0+1)}.$$

Before we prove Theorem 3.3, we need to introduce the following terminology. For $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0$, define the set

$$\mathcal{H}(n, \alpha) := \{[(u_1, s_1), \ldots, (u_n, s_n)] \in (\mathbb{N}_0 \times \mathbb{N}_0)^n : \ u_1 + s_1 + \ldots + u_n + s_n = \alpha\}. \tag{3.9}$$

and denote by $P(n, \alpha)$ the number of elements of $\mathcal{H}(n, \alpha)$, i.e., $P(n, \alpha) := \#\mathcal{H}(n, \alpha)$. Naturally, for each element $[(u_1, s_1), \ldots, (u_n, s_n)]$ of $\mathcal{H}(n, \alpha)$ we associate the chain $\mathfrak{b}_n = (u_1, s_1)\cdots(u_n, s_n)$ and, if $\mathfrak{b}_n$ is admissible, the n-return map $\Gamma_t^{\mathfrak{b}_n}$. If $n = 1$, we have

$$\begin{aligned} P(1, \alpha) &= \#\mathcal{H}(1, \alpha) \\ &= \#\{(u, s) : u + s = \alpha\} \\ &= \alpha + 1, \end{aligned} \tag{3.10}$$

and consequently there are at most $\alpha+1$ 1–return maps, $\Gamma_t^{(u,s)}$, such that $u+s=\alpha$.

Note that we can write $\mathcal{H}(n,\alpha)$ as the disjoint union

$$\mathcal{H}(n,\alpha)=\bigcup_{i=0}^{\alpha}\mathcal{H}(1,i)\times\mathcal{H}(n-1,\alpha-i),$$

so the cardinality of $\mathcal{H}(n,\alpha)$ is equal to

$$\begin{aligned} P(n,\alpha) &= \sum_{i=0}^{\alpha}P(1,i)\cdot P(n-1,\alpha-i) \\ &= \sum_{i=0}^{\alpha}(i+1)P(n-1,\alpha-i) \end{aligned} \tag{3.11}$$

since (3.10) holds.

Before we measure the growth of $\#\mathbb{P}_m(\mathfrak{F}_t)$, we need to introduce the following definition and state a technical result.

Definition 3.1. *Given $n\in\mathbb{N}$, we say that n verifies the* property $\mathcal{P}$ *if*

$$P(1,\alpha_1+1)+\cdots+P(n,\alpha_n+1)<2\Big(P(1,\alpha_1)+\cdots+P(n,\alpha_n)\Big),$$

for every $(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n$ such that $\alpha_i-\alpha_{i+1}>2$, for each $i=1,\ldots,n-1$.

Lemma 3.4. *The property $\mathcal{P}$ holds for all $n\in\mathbb{N}$.*

With the Lemma 3.4 (whose proof is done below), we are now in position to prove that $\#\mathbb{P}_m(\mathfrak{F}_t)$ grows at most exponentially fast.

Proof of Theorem 3.3. If $m=n_0+2$, then, for $(\mathfrak{b},x)\in\mathbb{P}_{n_0+2}(\mathfrak{F}_t)$, we have $\Gamma_t^{\mathfrak{b}}(x)=x$ with $\mathfrak{b}=(0,0)$ and, from Lemma 3.1, the map $\Gamma_t^{(0,0)}$ has at most two fixed points. For $m>n_0+2$, the number m can be written in the following ways: defining

$$\kappa=\kappa_m:=[m/(n_0+2)] \tag{3.12}$$

where $[\cdot]$ denotes the characteristic, we have

$$m=\kappa(n_0+2)+r_\kappa$$

with $0\le r_\kappa<n_0+2$; we can also write, for each $l=1,\ldots,\kappa-1$,

$$m=l(n_0+2)+r_l,$$

but in this cases we have $r_l\ge n_0+2$. In this way, for each $l\in\{1,\ldots,\kappa\}$ and for each chain $\mathfrak{b}_l=(u_1,s_1)\cdots(u_l,s_l)$ such that $|\theta(\mathfrak{b}_l)|=m$, that is,

$$m=|\theta(\mathfrak{b}_l)|=|0^{n_0+1+s_1}10^{u_1}\cdots0^{n_0+1+s_l}10^{u_l}|=l(n_0+2)+\sum_{i=1}^{l}(s_i+u_i),$$

we have

$$m=l(n_0+2)+r_l,\ \text{with}\ r_l=\sum_{i=1}^{l}(s_i+u_i).$$

Thus, see (3.9), for each $l\in\{1,\ldots,\kappa\}$,

$$P(l,r_l)=\#\mathcal{H}(l,r_l)$$

is the number of l-return maps $\Gamma_t^{\mathfrak{b}_l}$, some of them with empty domain, with $|\theta(\mathfrak{b}_l)| = m$.

Now, if $(\mathfrak{b}_l, x) \in \mathbb{P}_m(\mathfrak{F}_t)$, then $\Gamma_t^{\mathfrak{b}_l}(x) = x$ for some admissible chain $\mathfrak{b}_l$, where $l \in \{1, \dots, \kappa\}$ and the chain $\mathfrak{b}_l$ gives the itinerary that the point x follows. Consequently

$$I_{\mathfrak{F}_t}(m) := P(1, r_1) + \cdots + P(\kappa, r_\kappa)$$

is the number of the possible itineraries with length m that a periodic point of the system $\mathfrak{F}_t$ can follows. As each l-return map $\Gamma_t^{\mathfrak{b}_l}$ has at most two fixed points, Lemma 3.1, we conclude that

$$\begin{aligned} \#\mathbb{P}_m(\mathfrak{F}_t) &\leq 2 \cdot I_{\mathfrak{F}_t}(m) \\ &= 2\Big(P(1, r_1) + P(2, r_2) + \cdots P(\kappa, r_\kappa)\Big), \end{aligned} \tag{3.13}$$

and the result follow immediately if we prove the following claim.

Claim. *For all $t > 0$ small enough and $m \geq n_0 + 2$ it holds $I_{\mathfrak{F}_t}(m) \leq 2^{m-(n_0+2)}$.*

Proof. We argue inductively on m.

If $m = n_0 + 2$, then $I_{\mathfrak{F}_t}(m) = P(1, 0) = 1$, corresponding to the 1-return map $\Gamma_t^{(0,0)}$. Assume that the claim is true for m, that is,

$$I_{\mathfrak{F}_t}(m) = P(1, r_1) + P(2, r_2) + \cdots + P(\kappa, r_\kappa) \leq 2^{m-(n_0+2)}.$$

with $\kappa = \kappa_m$ defined by (3.12) and $r_\kappa = m - \kappa(n_0 + 2) \in \{0, 1, \dots, n_0 + 1\}$. To prove the estimate

$$I_{\mathfrak{F}_t}(m + 1) \leq 2^{m+1-(n_0+2)},$$

we separate the cases of $r_\kappa < n_0 + 1$ and $r_\kappa = n_0 + 1$:

Case 1: $r_\kappa < n_0 + 1$

In this case we have $\kappa = \kappa_m = \kappa_{m+1}$ (see (3.12)) and the possibilities for "the splitting of $m + 1$" are

$$m + 1 = l(n_0 + 2) + r_l + 1, \; l = 1, \dots, \kappa,$$

that is we have the same combinatory as the one we had for m. Since $r_l - r_{l+1} = n_0 + 2 > 2$, for each $l = 1, \dots, \kappa - 1$, by Lemma 3.4,

$$\begin{aligned} I_{\mathfrak{F}_t}(m + 1) &= P(1, r_1 + 1) + \cdots + P(\kappa, r_\kappa + 1) \\ &< 2\left(P(1, r_1) + \cdots + P(\kappa, r_\kappa)\right), \end{aligned} \tag{3.14}$$

and, by induction, we conclude that $I_{\mathfrak{F}_t}(m + 1) \leq 2 \times 2^{m-(n_0+2)}$.

Case 2: $r_\kappa = n_0 + 1$

In this case, the possibilities for the splitting of $m + 1$ are

$$\begin{cases} m + 1 = l(n_0 + 2) + r_l + 1, \; l = 1, \; \dots, \kappa \\ m + 1 = (\kappa + 1)(n_0 + 2), \end{cases},$$

and consequently

$$\begin{aligned} I_{\mathfrak{F}_t}(m + 1) &= P(1, r_1 + 1) + \cdots + P(\kappa, r_\kappa + 1) + P(\kappa + 1, 0) \\ &= P(1, r_1 + 1) + \cdots + P(\kappa, r_\kappa + 1) + 1 \\ &< 2\left(P(1, r_1) + \cdots + P(\kappa, r_\kappa)\right) + 1 \\ &\leq 2\left(P(1, r_1) + \cdots + P(\kappa, r_\kappa)\right), \end{aligned} \tag{3.15}$$

where the first inequality follows from Lemma 3.4 again.

By induction hypothesis, from (3.14) and (3.15) we have

$$I_{\mathfrak{F}_t}(m + 1) \leq 2 \cdot I_{\mathfrak{F}_t}(m) \leq 2 \cdot 2^{m-(n_0+2)} = 2^{m+1-(n_0+2)},$$

and the claim is proved. □

It remains to prove Lemma 3.4. To prove this lemma we use complete induction on n.

Proof of Lemma 3.4. We shall show by complete induction that the property $\mathcal{P}$ holds for all $n \in \mathbb{N}$.
As

$$P(1, \alpha + 1) = \alpha + 2 < 2(\alpha + 1) = 2P(1, \alpha),\ \forall \alpha \in \mathbb{N},$$

then the property $\mathcal{P}$ is trivially verified for $n = 1$.

Now, fix $n \in \mathbb{N}$ and assume that the property $\mathcal{P}$ holds for all natural number m less than n. We shall show that the property $\mathcal{P}$ also holds for n. Taking a sequence $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ verifying

$$\alpha_i - \alpha_{i+1} > 2, \quad \text{for each } i = 1, \ldots, n-1, \tag{3.16}$$

the goal is to prove that

$$P(1, \alpha_1 + 1) + P(2, \alpha_2 + 1) + \cdots + P(n, \alpha_n + 1) < 2\Big(P(1, \alpha_1) + P(2, \alpha_2) + \cdots + P(n, \alpha_n)\Big).$$

Applying (3.11) to $P(2, \alpha_2 + 1), \cdots, P(n, \alpha_n + 1)$, we have

$$\begin{aligned} & P(1, \alpha_1 + 1) + P(2, \alpha_2 + 1) + \cdots + P(n, \alpha_n + 1) = \\ = \; & P(1, \alpha_1 + 1) + \Big(P(1, \alpha_2 + 1) + 2P(1, \alpha_2) + \cdots + (\alpha_2 + 2)P(1, 0)\Big) + \\ & + \cdots + \Big(P(n-1, \alpha_n + 1) + 2P(n-1, \alpha_n) + \cdots + (\alpha_n + 2)P(n-1, 0)\Big) \end{aligned}$$

Noting that (3.16) implies that $\alpha_n + 2(n-1) < \alpha_{n-1} + 2(n-2) < \cdots < \alpha_2 + 2 < \alpha_1$, we may reorganizing the previous sum in the following way

$$\begin{aligned} & P(1, \alpha_1 + 1) + \cdots + P(n, \alpha_n + 1) = \\ = \; & P(1, \alpha_1 + 1) + \Big(P(1, \alpha_2 + 1) + P(2, \alpha_3 + 1) + \cdots + P(n-1, \alpha_n + 1)\Big) + \\ & + 2\Big(P(1, \alpha_2) + P(2, \alpha_3) + \cdots + P(n-1, \alpha_n)\Big) + \cdots + \\ & + \alpha_n\Big(P(1, \alpha_2 - \alpha_n + 2) + P(2, \alpha_3 - \alpha_n + 2) + \cdots + P(l-1, 2)\Big) \\ & + (\alpha_n + 1)\Big(P(1, \alpha_2 - \alpha_n + 1) + P(2, \alpha_3 - \alpha_n + 1) + \cdots + P(n-1, 1)\Big) + \\ & + (\alpha_n + 2)\Big(P(1, \alpha_2 - \alpha_n) + P(2, \alpha_3 - \alpha_n) + \cdots + P(n-2, 1) + 1\Big) + \\ & + \cdots + \alpha_2 P(1, 2) + (\alpha_2 + 1)P(1, 1) + (\alpha_2 + 2)P(1, 0). \end{aligned}$$

Observe that $P(n-i, 0) = 1$ for all $i \in \{1, \cdots, n-\}$. Applying the induction hypothesis, we have

$$\begin{aligned} & P(1, \alpha_1 + 1) + P(2, \alpha_2 + 1) + \cdots + P(n, \alpha_n + 1) < \\ < \; & P(1, \alpha_1 + 1) + 2\Big(P(1, \alpha_2) + P(2, \alpha_3) + \cdots + P(n-1, \alpha_n)\Big) + \\ & + 2^2\Big(P(1, \alpha_2 - 1) + P(2, \alpha_3 - 1) + \cdots + P(n-1, \alpha_n - 1)\Big) + \cdots + \\ & + 2\alpha_n\Big(P(1, \alpha_2 - \alpha_n + 1) + P(2, \alpha_3 - \alpha_n + 1) + \cdots + P(n-1, 1)\Big) \\ & + 2(\alpha_n + 1)\Big(P(1, \alpha_2 - \alpha_n) + P(2, \alpha_3 - \alpha_n) + \cdots + P(n-1, 0)\Big) + \\ & + 2(\alpha_n + 2)\Big(P(1, \alpha_2 - \alpha_n - 1) + P(2, \alpha_3 - \alpha_n - 1) + \cdots + P(n-2, 0)\Big) + \\ & + \cdots + 2(\alpha_2)P(1, 1) + 2(\alpha_2 + 1)P(1, 0) + \alpha_2 + 2 \end{aligned}$$

and reorganizing again the terms of the sum, we obtain

$$\begin{aligned} & P(1, \alpha_1 + 1) + P(2, \alpha_2 + 1) + \cdots + P(n, \alpha_n + 1) < \\ < \; & P(1, \alpha_1 + 1) + \alpha_2 + 2 + 2\Big(P(1, \alpha_2) + 2P(1, \alpha_2 - 1) + \cdots + (\alpha_2 + 1)P(1, 0)\Big) \\ & + 2\Big(P(2, \alpha_3) + 2P(2, \alpha_3 - 1) + 3P(2, \alpha_3 - 2) + \cdots + (\alpha_3 + 1)P(2, 0)\Big) + \cdots + \\ & + 2\Big(P(n-1, \alpha_n) + 2P(n-1, \alpha_n - 1) + \cdots + (\alpha_n + 1)P(n-1, 0)\Big). \end{aligned}$$

Recalling that

$$P(i,\alpha_i) = 1P(i-1,\alpha_i) + 2P(i-1,\alpha_i-1) + \cdots (\alpha_i+1)P(i-1,0),\ i=1,\ldots,n,$$

$P(1,\alpha_1+1) = \alpha_1+2$, and $\alpha_2 < \alpha_1 - 2$, we obtain

$$\begin{array}{rl} & P(1,\alpha_1+1) + P(2,\alpha_2+1) + \cdots + P(n,\alpha_n+1) < \\ < & P(1,\alpha_1+1) + \alpha_2 + 2 + 2P(2,\alpha_2) + 2P(3,\alpha_3) + \cdots + 2P(n,\alpha_n) \\ = & P(1,\alpha_1+1) + \alpha_2 + 2 + 2\Big(P(2,\alpha_2) + \cdots + P(n,\alpha_n)\Big) \\ = & \alpha_1 + 2 + \alpha_2 + 2 + 2\Big(P(2,\alpha_2) + \cdots + P(n,\alpha_n)\Big) \\ < & 2\,(\alpha_1+1) + 2\Big(P(2,\alpha_2) + \cdots + P(n,\alpha_n)\Big), \end{array}$$

which implies that

$$\begin{array}{rl} & P(1,\alpha_1+1) + P(2,\alpha_2+1) + \cdots + P(n,\alpha_n+1) \\ < & 2P(1,\alpha_1) + 2\Big(P(2,\alpha_2) + \cdots + P(n,\alpha_n)\Big) \\ = & 2\Big(P(1,\alpha_1) + P(2,\alpha_2) + \cdots + P(n,\alpha_n)\Big), \end{array}$$

Consequently the property $\mathcal{P}$ holds for n as required, ending the proof of the lemma. □

4 Dynamics in a neighborhood of the cycle. Prove the main result

In this section, we build the convenient neighborhood of the cycle of G_0, $\mathcal{V}$, associated to the fixed points $P = (0^{\mathbb{Z}}, 1/2)$ and $Q = (0^{\mathbb{Z}}, 0)$ in order to prove Theorem 1.1.

First, we recall that the heteroclinic set $\mathcal{I}_{P,Q} := \{0^{\mathbb{Z}}\} \times (0, 1/2)$ satisfies (see (1.2))

$$\mathcal{I}_{P,Q} \subseteq W^u(Q, G_t) \cap W^s(P, G_t)$$

for all $t \geq 0$. Given $0 < \varepsilon < \epsilon$ (see definition of ϵ in condition (h1)) and $k \in \mathbb{N}$, we define the (k,ε)-neighborhood of $\mathcal{I}_{P,Q}$ as the set

$$V(\mathcal{I}_{P,Q}, k, \varepsilon) := \left[0^{-k}.0^{k}\right] \times \left(-\varepsilon, \frac{1}{2} + \varepsilon\right), \tag{4.1}$$

where $[0^{-k}.0^k] = \{(\xi_i)_{i\in\mathbb{Z}} \in \Sigma_2 :\ \xi_i = 0 \text{ for } i \in \{-k,\ldots,-1,0,1,\ldots,k-1\}\}$ (see section 2).

It is easy to verify that the point

$$Z := \left(0^{-\mathbb{N}}.10^{\mathbb{N}}, \frac{1}{2}\right) \in W^u(P, G_0) \cap W^s(Q, G_0)$$

satisfies $G_0^{k+1+i}(Z) \in V(\mathcal{I}_{P,Q}, k, \varepsilon)$ and $G_0^{-k-i}(Z) \in V(\mathcal{I}_{P,Q}, k, \varepsilon)$, for all $i \in \mathbb{N}_0$, and we define a neighborhood of the point Z as the set

$$V(Z, k, \gamma) := \left[0^{-2k}.10^{2k}\right] \times \left(\frac{1}{2} - \gamma, \frac{1}{2} + \gamma\right) \tag{4.2}$$

where $\gamma \in (0, \varepsilon)$ such that

$$g_0^{-k}([1/2-\gamma, 1/2+\gamma]) \bigcup \left(g_0^{k+1} \circ g_{1,t}([1/2-\gamma, 1/2+\gamma])\right) \subseteq (-\varepsilon, 1/2+\varepsilon)\,. \tag{4.3}$$

By definitions of $V(Z,k,\gamma)$ and $V(\mathcal{I}_{P,Q}, k, \varepsilon)$ we have

$$G_0^{k+1}\left(V(Z,k,\gamma)\right) \cup G_0^{-k}(V(Z,k,\gamma)) \subseteq V(\mathcal{I}_{P,Q}, k, \varepsilon),$$

thus the set

$$\mathcal{V}(k,\varepsilon,\gamma) := V(\mathcal{I}_{P,Q},k,\varepsilon)\bigcup\left(\bigcup_{i=-k+1}^{k} G_0^i(V(Z,k,\gamma))\right) \tag{4.4}$$

is a neighborhood of the cycle of G_0 associated to the fixed points P and Q and we call it the (k,ε,γ)-neighborhood of the cycle.

For $t_0>0$ small enough, we can choose $\varepsilon>0$ and $\gamma\in(0,\varepsilon)$ such that

$$g_{1,t}^2(1/2) = 2t-1/2 < t+d_t-1/2 < -\varepsilon,\ \forall t\in[0,t_0]$$

and (4.3) holds. Now we want to study the maximal invariant set of G_t in $\mathcal{V}(k,\varepsilon,\gamma)$,

$$\Lambda_t := \Lambda_t(k,\varepsilon,\gamma) = \bigcap_{i\in\mathbb{Z}} G_t^i(\mathcal{V}(k,\varepsilon,\gamma)),\ \forall t\in[0,t_0]. \tag{4.5}$$

Since k, ε and γ are fixed constants, for simplicity, we write $V(\mathcal{I}_{P,Q})$, $V(Z)$ and $\mathcal{V}$ instead of $V(\mathcal{I}_{P,Q},k,\varepsilon)$, $V(Z,k,\gamma)$ and $\mathcal{V}(k,\varepsilon,\gamma)$, respectively. Now, recall the definition of the fundamental domain $D_t=(g_0^{-1}(d_t),d_t]\subseteq(g_0^{-1}(t),1/2-t-\varepsilon)$ of g_0, (3.1), and consider the *cube* Δ_t defined by

$$\Delta_t := \{X=(\xi,x)\in\mathcal{V} : x\in D_t\}.$$

In order to study the relative dynamics of G_t on $\mathcal{V}$, we analyse the returns by G_t of points in the cube Δ_t to itself.

Definition 4.1. *Given $X\in\Delta_t$, we define the sequence of return times $(\varrho_i(X))_{i\in I(X)}$ of X to Δ_t by*

- $\varrho_0(X)=0$
- $\varrho_i(X)<\varrho_{i+1}(X)$, *$G_t^{\varrho_i(X)}(X)\in\Delta_t$ for all $i\in I(X)$, and $G_t^j(X)\notin\Delta_t$ for each $\varrho_i(X)<j<\varrho_{i+1}(X)$,*

where $I(X)$ is a (maximal) interval of $\mathbb{Z}$ containing 0 (this interval may be upper or/and lower bound). We denote by $X_{[i]}$ the i-th return of X to Δ_t.

The following Lemma, whose proof is omitted here (for details [DER, Propositions 7.5 and 7.10]), assures that all points on $\Lambda_t\setminus\{P,Q\}$ have some iterate in Δ_t and associates an itinerary to each point of Δ_t having returns to Δ_t.

Lemma 4.1. *Let $t\in[0,t_0]$. Then every $X\in\Lambda_t\setminus\{P,Q\}$ has some iterate, by G_t, in Δ_t and if $X=(\xi,x)\in\Delta_t$ has a sequence of forward return times $\varrho_i(X)$ to Δ_t, then*

$$\xi_0\xi_1\cdots\xi_{\varrho_i(X)-1} = 0^{n_0+1+s_1}10^{u_1+n_0+1+s_2}10^{u_2+n_0+1+s_3}1\cdots10^{u_{i-1}+n_0+1+s_i}10^{u_i},$$

where $u_j,\ s_j\in\mathbb{N}_0$ for every $j=1,\ldots,i$.

If $I(X)=\{0,\cdots,i\}$ and $X\in\Lambda_t$, then

$$\xi = \cdots\xi_{-1}.0^{n_0+1+s_1}10^{u_1+n_0+1+s_2}10^{u_2+n_0+1+s_3}1\cdots10^{u_{i-1}+n_0+1+s_i}10^{\mathbb{N}}.$$

From previous result, for $X=(\xi,x)\in\Delta_t$ and $(\varrho_i(X))_{i\in I(X)}$ the sequence of return times of X by G_t to Δ_t, we conclude that, for each $i\in I(X)$, the i-th return of X, $X_{[i]}=(\eta,x_{[i]})$, satisfies

$$\eta=\sigma^{\varrho_i(X)}(\xi) \text{ and } x_{[i]}=\Gamma_t^{\mathfrak{b}}(x),$$

with the chain $\mathfrak{b}=(u_1,s_1)\cdots(u_i,s_i)$. Consequently, if $X=(\xi,x)\in\Delta_t$ is a periodic point of G_t with period m, then

$$\xi=(0^{n_0+1+s_1}10^{u_1+n_0+1+s_2}1\cdots10^{u_{i-1}+n_0+1+s_i}10^{u_i})^{\mathbb{Z}} \text{ and } x=\Gamma_t^{\mathfrak{b}}(x) \tag{4.6}$$

for some $i\in\mathbb{N}$ and $\mathfrak{b}=(u_1,s_1)\cdots(u_i,s_i)$ such that $m\le i(n_0+2)+\sum_{j=1}^{i}(s_j+u_j)$.

Proposition 4.2. *Let $t \in [0, t_0]$. For each $m \geq n_0 + 2$, we have*

$$\#\mathbb{P}_m(G_t|_{\Lambda_t(\mathcal{V})}) \leq m \cdot \# (\mathbb{P}_m(\mathfrak{F}_t)) + 2$$

Proof. From Lemma 4.1, for each $X = (\xi, x) \in \mathbb{P}_m(G_t|_{\Lambda_t(\mathcal{V})}) \setminus \{P, Q\}$ there exists $r = r(X) \in \{0, \ldots, m-1\}$ such that

$$G_t^{r(X)}(X) \in \Delta_t \text{ and } G_t^j(X) \notin \Delta_t, \ \forall j \in \{0, \ldots, r-1\},$$

and, as $G_t^{r(X)}(X) \in \Delta_t$ is a periodic point of G_t with period m, by (4.6), we have

$$\sigma^{r(X)}(\xi) = \left(0^{n_0+1+s_1} 1 0^{u_1+n_0+1+s_2} 1 \cdots 0^{u_{l-1}+n_0+1+s_l} 1 0^{u_l}\right)^{\mathbb{Z}}$$

for some $l \in \mathbb{N}$ and $\mathfrak{b}(X) := (u_1, s_1) \cdots (u_l, s_l)$. Now, defining the map

$$\begin{array}{rccl} h_m: & \mathbb{P}_m(G_t|_{\Lambda_t(\mathcal{V})}) \setminus \{P, Q\} & \to & \mathbb{P}_m(\mathfrak{F}_t) \\ & X = (\xi, x) & \to & \left(\mathfrak{b}(X), g_{[\xi_0 \cdots \xi_{r(X)-1}]}(x)\right). \end{array}$$

The proposition follows from the next claim. The claim says that, if two periodic points of $G_t|_{\Lambda_t(\mathcal{V})}$ of period m have the same image by h_m, then they belong to the same orbit, that is:

Claim 4.3. *For every $X, Y \in \mathbb{P}_m(G_t|_{\Lambda_t(\mathcal{V})}) \setminus \{P, Q\} : h(X) = h(Y) \Rightarrow X = G_t^j(Y)$, for some $j \in \{0, \ldots, m-1\}$.*

Proof. Let $X = (\xi, x)$, $Y = (\xi^\star, y) \in \mathbb{P}_m(G_t|_{\Lambda_t(\mathcal{V})}) \setminus \{P, Q\}$ such that $h_m(X) = h_m(Y)$. By definition, we have

$$\sigma^{r(X)}(\xi) = \sigma^{r(Y)}(\xi^\star) \text{ and } g_{[\xi_0 \cdots \xi_{r(X)-1}]}(x) = g_{[\xi_0^\star \cdots \xi_{r(Y)-1}^\star]}(y)$$

Without losing generality, suppose that $r(X) \leq r(Y)$. Then $\xi = \sigma^{r(Y)-r(X)}(\xi^\star)$ and

$$\begin{aligned} x &= g^{-1}_{[\xi_0 \cdots \xi_{r(X)-1}]} \circ g_{[\xi_0^\star \cdots \xi_{r(Y)-1}^\star]}(y) \\ &= g^{-1}_{[\xi^\star_{r(Y)-r(X)} \cdots \xi^\star_{r(Y)-1}]} \circ g_{[\xi_0^\star \cdots \xi_{r(Y)-1}^\star]}(y) \\ &= g_{[\xi_0^\star \cdots \xi^\star_{r(Y)-r(X)-1}]}(y). \end{aligned}$$

This means that

$$X = (\xi, x) = \left(\sigma^{r(Y)-r(X)}(\xi^\star), g_{[\xi_0^\star \cdots \xi^\star_{r(Y)-r(X)-1}]}(y)\right) = G_t^{r(Y)-r(X)}(\xi^\star, y) = G_t^{r(Y)-r(X)}(Y)$$

and the claim is proved. □

Now the proof is complete. □

Finally, we present an example of a two-parameter family of skew product maps $G_{a,t}$ maps satisfying $H_{\mathcal{V}}(P, G_{a,t}) = H_{\mathcal{V}}(Q, G_{a,t})$.

Example 4.4. *Consider a two-parameter family of skew product maps $G_{a,t} : \Sigma_2 \times (-1/(2(e^a - 1)), 1] \to \Sigma_2 \times \mathbb{R}$, with $t \in [-1, 1]$ and $a > 0$, such that*

$$g_0(x) = g_a(x) = \frac{xe^a}{2xe^a + (1 - 2x)} \text{ and } g_{1,t,a}(x) = g_{1,t}(x) = x - 1/2 + t.$$

The map g_a has two fixed points in $(-1/(2(e^a - 1)), 1]$, 0 and $1/2$, and, for every $x \in [0, 1/2]$ and $n \in \mathbb{Z}$, we have

$$g_a^n(x) = \frac{xe^{na}}{2xe^{na} + (1 - 2x)}, \tag{4.7}$$

which, naturally, is a differentiable function with

$$(g_a^n)'(x) = \frac{e^{-na}}{x^2} \left(g_a^n(x)\right)^2, \; x \neq 0, \quad \textit{and } (g_a^n)'(0) = e^{na}. \tag{4.8}$$

In particular, we have $g_a'(0) = e^a$ *and* $g_a'(1/2) = e^{-a}$.

In [DER] the authors proved that this family, for $a \in (0, \log 2)$ *and* t *sufficiently small, satisfies the (EC) condition and, due an analytic symmetric property, we obtained that the relative homoclinic classes are equal, that is,* $H_{\mathcal{V}}(P, G_{a,t}) = H_{\mathcal{V}}(Q, G_{a,t})$.

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