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GOLDEN TILINGS

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ABSTRACT. We introduce the notion of golden tilings, and we prove a one-to-one correspondence between (i) smooth conjugacy classes of Anosov diffeomorphisms, with an invariant measure absolutely continuous with respect to the Lebesgue measure, (ii) affine classes of golden tilings and (iii) solenoid functions. The solenoid functions give a parametrization of the infinite dimensional space consisting of the mathematical objects described in the above equivalences.

1. Introduction

Inspired in the works of Y. Jiang [12], A. Pinto and D. Sullivan [28] and A. Pinto and D. Rand [19], we introduce the notion of *golden tilings*. The golden tilings record the infinitesimal geometric structure determined by the dynamics along the unstable leaf that is invariant by the Anosov diffeomorphism. We define the properties of the golden tilings using the Fibonacci decomposition of the natural numbers instead of the dyadic decomposition, because the Fibonacci decomposition has the advantage of encoding, in a natural way, the combinatorics determined by the Markov partition along the unstable leaf. Our goal is to exhibit a natural correspondence between golden tilings, Anosov diffeomorphisms and solenoid functions.

1.1. **Main theorem.** Let $\gamma = 2/(1+\sqrt{5})$ be the inverse of the golden number. The *(golden) Anosov automorphism* $G_A : \mathbb{T} \to \mathbb{T}$ is given by $G_A(x, y) = (x + y, x)$, where $\mathbb{T}$ is equal to $\mathbb{R}^2/(v\mathbb{Z} \times w\mathbb{Z})$ with $v = (\gamma, 1)$ and $w = (-1, \gamma)$. A C^{1+} *(golden) Anosov diffeomorphism* $G : \mathbb{T} \to \mathbb{T}$ is a $C^{1+\alpha}$ diffeomorphism, with $\alpha > 0$, such that (i) G is topologically conjugate to G_A by a map h_G; (ii) the tangent bundle has a $C^{1+\alpha}$ uniformly hyperbolic splitting into a stable direction and an unstable direction (see [30]). We denote by $\mathcal{G}$ the set of all such C^{1+} Anosov diffeomorphisms with an invariant measure absolutely continuous with respect to the Lebesgue measure.

The eigenvalues of the Anosov automorphism G_A are $\mu^- = -\gamma$ and $\mu^+ = 1/\gamma$. Let $\pi : \mathbb{R}^2 \to \mathbb{T}$ be the natural projection of $\mathbb{R}^2$ in $\mathbb{T}$. Let $\mathbf{A}_0$ and $\mathbf{B}_0$ be the rectangles $[0, 1] \times [0, 1]$ and $[-\gamma, 0] \times [0, \gamma]$, respectively (see Figure 1). A Markov partition $\mathcal{M}_A$ of G_A is given by $\mathbf{A} = \pi(\mathbf{A}_0)$ and $\mathbf{B} = \pi(\mathbf{B}_0)$. The unstable manifolds of G_A correspond to the projection by π of the vertical lines of the plane, and the stable manifolds of G_A are the projection by π of the horizontal lines of the plane. Let W_0 be the positive vertical axis of $\mathbb{R}^2$. Hence $W = \pi(W_0)$ is the unstable leaf of G_A with only one endpoint $y_0 = \pi(0, 0)$ that is the fixed point of G_A. The unstable

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leaf W passes first through all the unstable boundaries of the Markov rectangles **A** and **B**. Let the unstable spanning leaf segment K_1 be the left unstable boundary of the Markov rectangle **A** (see the definition of a spanning leaf segment in Section 2.1). Let the unstable spanning leaf segment K_2 be the left unstable boundary of the Markov rectangle **B**. Let $K_3, K_4, \ldots \in W$ be the unstable leaf segments defined, inductively, as follows: (i) K_i is an unstable spanning leaf of a Markov rectangle, for every $i \geq 3$; (ii) $K_i \cap K_{i+1} = \{y_i\}$ is a common boundary point of both K_i and K_{i+1}, for every $i \geq 2$ (see Figure 1). We note that $W = \bigcup_{i \geq 1} K_i$.

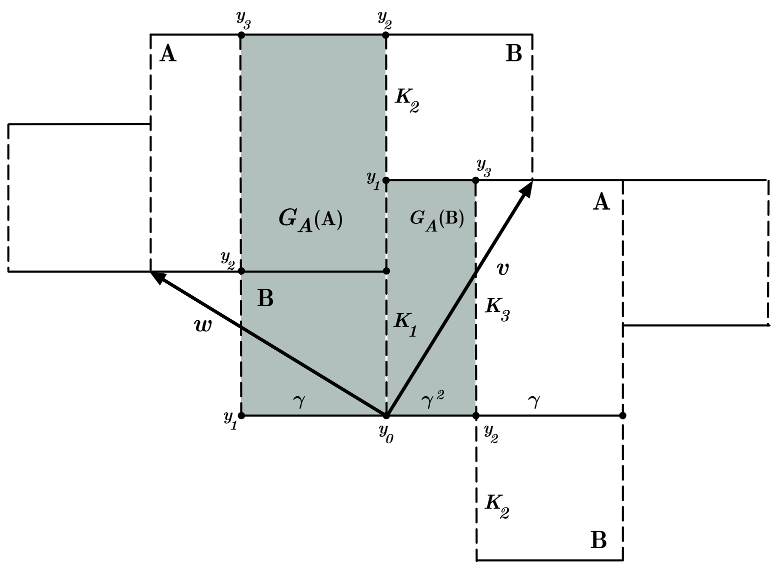


FIGURE 1. The golden automorphism G_A.

Theorem 1.1 (Flexibility). *There is a well-defined map $G \to (a_i^G)_{i\in\mathbb{N}}$ that associates to each C^{1+} Anosov diffeomorphism G in $\mathcal{G}$ the golden sequence $(a_i^G)_{i\in\mathbb{N}}$ given by*

$$a_i^G \quad = \quad \lim_{n\to\infty} \frac{|G^{-n}(h_G(K_{i+1}))|}{|G^{-n}(h_G(K_i))|},$$

where $|I|$ means the length of the unstable leaf segment I with respect to a Riemannian metric on $\mathbb{T}$. Furthermore, this map determines a one-to-one correspondence between smooth conjugacy classes of Anosov diffeomorphisms in $\mathcal{G}$ and golden sequences.

We leave the definition of the golden sequences to Section 3.7, and we will prove Theorem 1.1 in Section 3.8.

The Fibonacci numbers $F_1, F_2, F_3, \ldots$ are inductively given by the well-known relation $F_{n+2} = F_{n+1} + F_n$, with $n \geq 1$, where we choose $F_1 = 1$ and $F_2 = 2$. For any natural number $i \in \mathbb{N}$, we define the finite sequence $F_{n_0}, F_{n_1}, \ldots, F_{n_p}$ as follows: (i) F_{n_0} is the largest Fibonacci less or equal to i; (ii) inductively, if $F_{n_0}+\cdots+F_{n_{k-1}} < i$, then F_{n_k} is the largest Fibonacci less than or equal to $i - \left(F_{n_0} + \cdots + F_{n_{k-1}}\right)$. We observe that there is an integer $p \in \mathbb{N}$ and a Fibonacci number F_{n_p} such that

$$i = F_{n_0} + \cdots + F_{n_p}.$$

We call $F_{n_0}, \dots, F_{n_p}$ the *Fibonacci decomposition* of $i \in \mathbb{N}$. We observe that every natural number $i \in \mathbb{N}$ has a unique Fibonacci decomposition.

Definition 1.2. The *rigid* golden sequence $(a_i^R)_{i\in\mathbb{N}}$ is defined as follows. For every $i \in \mathbb{N}$, with Fibonacci decomposition $F_{n_0}, \dots, F_{n_p}$, we define

(i) $a_i^R = \gamma^{-1}$ if either ($n_p = 1$ and n_{p-1} is odd) or ($n_p = 2$ and n_{p-1} is even);
(ii) $a_i^R = \gamma$ if either ($n_p = 1$ and n_{p-1} is even) or ($n_p > 2$ and n_p is odd);
(iii) $a_i^R = 1$ if either ($n_p = 2$ and n_{p-1} is odd) or ($n_p > 2$ and n_p is even).

In Theorem 1.1, we prove the existence of an infinite dimensional space of well-characterized golden sequences. However, the only golden sequence that we are able to make explicit is the rigid golden sequence.

Theorem 1.3 (Rigidity)**.** *Every Anosov diffeomorphism $G \in \mathcal{G}$, with a $C^{1+zygmund}$ complete system of unstable holonomies, determines a golden sequence $(a_i^G)_{i\in\mathbb{N}}$ that is rigid.*

The definition of a $C^{1+zygmund}$ *complete system of unstable holonomies* and the proof of Theorem 1.3 are given in Section 4.

2. Solenoid functions

Let G be a C^{1+} *Anosov diffeomorphism* in $\mathcal{G}$. We define the map $G_\iota = G$ if $\iota = u$ or $G_\iota = G^{-1}$ if $\iota = s$. For $\iota \in \{s, u\}$ and $x \in \mathbb{T}$, we denote the local ι-manifolds through x by

$$W^\iota(x, \varepsilon) = \left\{ y \in \mathbb{T} : d(G_\iota^{-n}(x), G_\iota^{-n}(y)) \le \varepsilon, \text{ for all } n \ge 0 \right\},$$

where d is the distance determined by a Riemannian metric on the torus. By the Stable Manifold Theorem [30], these sets are respectively contained in the stable and unstable immersed manifolds

$$W^\iota(x) = \bigcup_{n\ge 0} G_\iota^n \left(W^\iota \left(G_\iota^{-n}(x), \varepsilon_0 \right) \right)$$

which are the image of $C^{1+\alpha}$ immersions $\kappa_{\iota,x} : \mathbb{R} \to \mathbb{T}$, for some $0 < \alpha \le 1$ and some small $\varepsilon_0 > 0$. An *open* (resp. *closed*) *ι-leaf segment* I is defined as a subset of $W^\iota(x)$ of the form $\kappa_{\iota,x}(I_1)$, where I_1 is an open (resp. closed) subinterval (non-empty) in $\mathbb{R}$. An *ι-leaf segment* is either an open or closed ι-leaf segment. The *endpoints* of an ι-leaf segment $I = \kappa_{\iota,x}(I_1)$ are the points $\kappa_{\iota,x}(u)$ and $\kappa_{\iota,x}(v)$, where u and v are the endpoints of I_1. The *interior* of an ι-leaf segment I is the complement of its boundary. A map $c : I \to \mathbb{R}$ is an *ι-leaf chart* of an ι-leaf segment I if c is a homeomorphism onto its image.

2.1. **Spanning leaf segments.** One can find a small enough $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$ there is $\delta = \delta(\epsilon) > 0$ with the property that, for all points $w, z \in \mathbb{T}$ with $d(w, z) < \delta$, $W^u(w, \epsilon)$ and $W^s(z, \epsilon)$ intersect in a unique point that we denote by

$$[z, w] = W^u(w, \epsilon) \cap W^s(z, \epsilon).$$

A rectangle R is a subset of $\mathbb{T}$ which is (i) closed under the bracket, i.e. $x, y \in R \Rightarrow [x, y] \in R$, and (ii) proper, i.e. it is the closure of its interior in $\mathbb{T}$. If ℓ^u and ℓ^s are, respectively, unstable and stable closed leaf segments intersecting in a single point, then we denote by $[\ell^u, \ell^s]$ the set consisting of all points of the form $[z, w]$ with $z \in \ell^u$ and $w \in \ell^s$. We note that $[\ell^u, \ell^s]$ is a rectangle. Conversely, given a rectangle R, for each $x \in R$ there are closed unstable and stable leaf segments of $\mathbb{T}$, $\ell^u(x,R) \subset W^u(x)$ and $\ell^s(x,R) \subset W^s(x)$ such that $R = [\ell^u(x,R), \ell^s(x,R)]$. The leaf segments $\ell^u(x,R)$ and $\ell^s(x,R)$ are called, respectively, *unstable* and *stable spanning leaf segments* (see Figure 2).

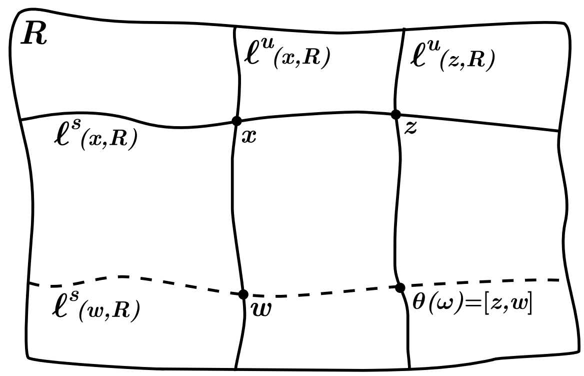


FIGURE 2. A basic unstable holonomy $\theta : \ell^u(x,R) \to \ell^u(z,R)$.

2.2. **Basic holonomies.** Suppose that x and z are two points inside any rectangle R of $\mathbb{T}$. Let $\ell^u(x,R)$ and $\ell^u(z,R)$ be two unstable spanning leaf segments of R containing, respectively, x and z. We define the map $\theta : \ell^u(x,R) \to \ell^u(z,R)$ by $\theta(w) = [z,w]$ (see Figure 2). Such maps are called the *basic unstable holonomies.* They generate the pseudo-group of all unstable holonomies. Similarly, we can define the *basic stable holonomies.*

2.3. **Lamination atlas.** The *unstable lamination atlas* $\mathcal{L} = \mathcal{L}^u(G,\rho)$*, determined by a Riemannian metric* ρ, is the set of all maps $e : I \to \mathbb{R}$, where e is an isometry between the induced Riemannian metric on the unstable leaf segment I and the Euclidean metric on the reals. We call the maps $e \in \mathcal{L}$ the *unstable lamination charts.* By Theorem 2.1 in [23], the basic unstable and stable holonomies are C^{1+} with respect to the lamination atlas $\mathcal{L}$ (see also [1] and [17]).

2.4. **HR structures.** The *HR structure* associates an affine structure to each stable and unstable leaf segment of G_A in such a way that these vary Hölder continuously with the leaf and are invariant under G_A (see [24]).

An affine structure on a stable or unstable leaf of G_A is equivalent to a *ratio function* $r(I:J)$ which can be thought of as prescribing the ratio of the size of two leaf segments I and J in the same stable or unstable leaf. A *ratio function* $r(I:J)$ is positive (we recall that each leaf segment has at least two distinct points) and continuous in the endpoints of I and J. Moreover,

$$r(I:J) = r(J:I)^{-1} \text{ and } r(I_1 \cup I_2 : K) = r(I_1:K) + r(I_2:K), \tag{2.1}$$

provided I_1 and I_2 intersect in at most one of their endpoints.

We say that r is an *unstable ratio function* if (i) for all unstable leaf segments K and for all unstable leaf segments $I, J \subset K$, $r(I:J)$ defines a ratio function on K; (ii) r is invariant under G_A, i.e. $r(I:J) = r(G_A(I):G_A(J))$ for all unstable leaf segments; and (iii) for every basic unstable holonomy map $\theta : I \to J$ between the unstable leaf segment I and the unstable leaf segment J, defined with respect to a rectangle R, and for every unstable leaf segment $I_0 \subset I$ and every unstable leaf segment $I_1 \subset I$,

$$\left| \log \frac{r(\theta I_0 : \theta I_1)}{r(I_0 : I_1)} \right| \leq \mathcal{O}\left((d(I,J))^{\epsilon} \right), \tag{2.2}$$

where $\epsilon \in (0,1)$ depends upon R and the constant of proportionality also depends upon R, but not on the segments considered. Since r satisfies condition (2.2) and defines an affine structure on each leaf that is invariant under G_A, we say that r is a *tranversely Hölder unstable ratio function*. An *HR-structure* is a pair (r^s, r^u) consisting of a stable and an unstable ratio function.

2.5. **Realized HR structures.** Let G be a C^{1+} Anosov diffeomorphism in $\mathcal{G}$, and let $\mathcal{L}^u(G,\rho)$ be an unstable lamination atlas associated to a Riemannian metric ρ. If I is an unstable leaf segment, then by $|I|$ we mean the length of the unstable leaf containing I measured using the Riemannian metric ρ. Let $h_G : \mathbb{T} \to \mathbb{T}$ be the topological conjugacy between the automorphism G_A and the Anosov diffeomorphism G. Using the mean value theorem and the fact that G is $C^{1+\alpha}$ uniformly hyperbolic, for some $\alpha > 0$, for all short unstable leaf segments K of G_A and all leaf segments I and J contained in K, the unstable realized ratio function r^u_G given by

$$r^u_G(I : J) \quad = \quad \lim_{n\to\infty} \frac{|G^{-n}(h_G(J))|}{|G^{-n}(h_G(I))|} \tag{2.3}$$

is well defined (see Lemma 3.6 in [24]). Similarly, we have the definition of the stable realized ratio function r^s_G.

Lemma 2.1. *The map $G \to r^u_G$ determines a one-to-one correspondence between C^{1+} conjugacy classes of Anosov diffeomorphisms G in $\mathcal{G}$ and unstable ratio functions.*

Proof. By Theorem 5.1 in [24], there is a one-to-one correspondence $G \to (r^s_G, r^u_G)$ between C^{1+} conjugacy classes of Anosov diffeomorphisms G and HR-structures (r^s_G, r^u_G). By Lemma 5 in [19], the unstable ratio function r^u_G determines, uniquely, the stable ratio function r^s_G if the Anosov diffeomorphism G has an invariant measure that is absolutely continuous with respect to the Lebesgue measure (see also [3]). □

2.6. **Realizable solenoid functions.** Let **sol** denote the set of all ordered pairs (I,J) of unstable spanning leaf segments of the Markov rectangles **A** and **B** of G_A such that the intersection of I and J consists of a single endpoint. Since the set **sol** is topologically a finite disjoint union of disjoint intervals, i.e. the disjoint union of a primary stable leaf of **A** and a primary stable leaf of **B**, it has a natural topological structure.

We define a pseudo-metric $d_{\mathbf{sol}} : \mathbf{sol} \times \mathbf{sol} \to \mathbb{R}^+$ on the set **sol** by

$$d_{\mathbf{sol}}\left((I,J),(I',J')\right) = \max\left\{d\left(I,I'\right), d\left(J,J'\right)\right\} \ .$$

Let G be a C^{1+} Anosov diffeomorphism in $\mathcal{G}$. We refer to the restriction $r^u_G|\mathbf{sol}$ of an unstable ratio function r^u_G to **sol** as the *unstable realized solenoid function*, and we denote it by $\sigma_G = r^u_G|\mathbf{sol}$. By construction, the restriction σ_G of the unstable ratio function to **sol** gives a Hölder continuous function satisfying the matching condition and the boundary condition, which we now proceed to describe (see Theorem 6.1 in [24]).

2.7. **Hölder continuity of solenoid functions.** The *Hölder continuity* of solenoid functions means that for all $t = (I, J)$ and $t' = (I', J')$ in **sol**,

$$|\sigma_G(t) - \sigma_G(t')| \leq \mathcal{O}\left((d_{\mathbf{sol}}(t, t'))^{\alpha}\right),$$

for some $\alpha > 0$.

2.8. **Matching condition.** Let $(I, J) \in \mathbf{sol}$. Suppose that there are pairs

$$(I_0, I_1), (I_1, I_2), \ldots, (I_{n-2}, I_{n-1}) \in \mathbf{sol}$$

such that $G_A I = \bigcup_{j=0}^{k-1} I_j$ and $G_A J = \bigcup_{j=k}^{n-1} I_j$. Then

$$\frac{|G_A J|}{|G_A I|} = \frac{\sum_{j=k}^{n-1} |I_j|}{\sum_{j=0}^{k-1} |I_j|} = \frac{\sum_{j=k}^{n-1} \prod_{i=1}^{j} |I_i|/|I_{i-1}|}{1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} |I_i|/|I_{i-1}|} .$$

Hence, the realized solenoid function σ_G must satisfy the *matching condition* (see Figure 3) for all such leaf segments:

$$\sigma_G(I : J) = \frac{\sum_{j=k}^{n-1} \prod_{i=1}^{j} \sigma_G(I_{i-1} : I_i)}{1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} \sigma_G(I_{i-1} : I_i)} . \tag{2.4}$$

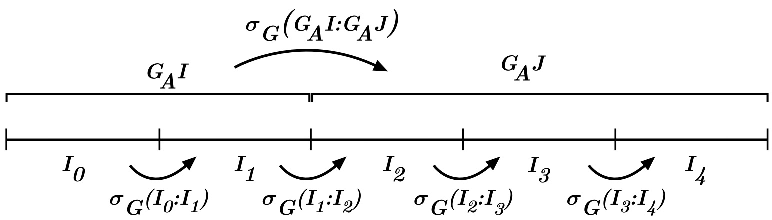


FIGURE 3. The matching condition for the solenoid function σ_G with $k = 2$ and $n = 5$.

Lemma 2.2. *Let $\sigma_G : \mathbf{sol} \to \mathbb{R}^+$ be a realized solenoid function. The matching condition holds for σ_G if, for every $(K_1, K_2) \in \mathbf{sol}$, the following conditions hold:*

(i) If $K_1, K_2 \in \mathbf{A}$, then

$$\sigma_G(K_1 : K_2) = \frac{\sigma_G(I_1 : I_2)\sigma_G(I_2 : I_3)\left(1 + \sigma_G(I_3 : I_4)\right)}{1 + \sigma_G(I_1 : I_2)}, \tag{2.5}$$

where I_1, I_2, I_3 and I_4 are such that $G_A(K_1) = I_1 \cup I_2$, $G_A(K_2) = I_3 \cup I_4$ and $(I_i, I_{i+1}) \in \mathbf{sol}$ for $i \in \{1, 2, 3\}$.

(ii) If $K_1 \in \mathbf{A}$ and $K_2 \in \mathbf{B}$, then

$$\sigma_G(K_1 : K_2) = \frac{\sigma_G(I_1 : I_2)\sigma_G(I_2 : I_3)}{1 + \sigma_G(I_1 : I_2)}, \tag{2.6}$$

where I_1, I_2 and I_3 are such that $G_A(K_1) = I_1 \cup I_2$, $G_A(K_2) = I_3$ and $(I_i, I_{i+1}) \in \mathbf{sol}$ for $i \in \{1, 2\}$.

(iii) If $K_1 \in \mathbf{B}$ and $K_2 \in \mathbf{A}$, then

$$\sigma_G(K_1 : K_2) = \sigma_G(I_1 : I_2)\left(1 + \sigma_G(I_2 : I_3)\right), \tag{2.7}$$

where I_1, I_2 and I_3 are such that $G_A(K_1) = I_1$, $G_A(K_2) = I_2 \cup I_3$ and $(I_i, I_{i+1}) \in \mathbf{sol}$ for $i \in \{1, 2\}$.

Proof. If $(K_1, K_2) \in \mathbf{sol}$, then (K_1, K_2) satisfies either condition (i), (ii) or (iii) above (see Figure 7). Let us check that the formulas (2.5), (2.6) and (2.7) correspond to the matching condition (2.4) for σ_G:

(i) If $K_1, K_2 \in \mathbf{A}$, then there exists $(I_i, I_{i+1}) \in \mathbf{sol}$, for $i = 1, 2, 3$, such that $G_A(K_1) = I_1 \cup I_2$ and $G_A(K_2) = I_3 \cup I_4$. Furthermore,

$$\frac{|G_A(K_2)|}{|G_A(K_1)|} = \frac{|I_3| + |I_4|}{|I_1| + |I_2|} = \frac{|I_2|}{|I_1|}\frac{|I_3|}{|I_2|}\left(1 + \frac{|I_4|}{|I_3|}\right)\left(1 + \frac{|I_2|}{|I_1|}\right)^{-1} .$$

Hence, equality (2.5) follows from equality (2.4).

(ii) If $K_1 \in \mathbf{A}$ and $K_2 \in \mathbf{B}$, then there exists $(I_i, I_{i+1}) \in \mathbf{sol}$, for $i = 1, 2$, such that $G_A(K_1) = I_1 \cup I_2$ and $G_A(K_2) = I_3 \cup I_4$. Furthermore,

$$\frac{|G_A(K_2)|}{|G_A(K_1)|} = \frac{|I_3|}{|I_1| + |I_2|} = \frac{|I_2|}{|I_1|}\frac{|I_3|}{|I_2|}\left(1 + \frac{|I_2|}{|I_1|}\right)^{-1} .$$

Hence, equality (2.6) follows from equality (2.4).

(iii) If $K_1 \in \mathbf{B}$ and $K_2 \in \mathbf{A}$, then there exists $(I_i, I_{i+1}) \in \mathbf{sol}$, for $i = 1, 2$, such that $G_A(K_1) = I_1$ and $G_A(K_2) = I_2 \cup I_3$. Furthermore,

$$\frac{|G_A(K_2)|}{|G_A(K_1)|} = \frac{|I_2| + |I_3|}{|I_1|} = \frac{|I_2|}{|I_1|}\left(1 + \frac{|I_3|}{|I_2|}\right) .$$

Hence, equality (2.7) follows from equality (2.4). □

2.9. **Boundary condition.** Let $(I_i, I_{i+1}), (J_j, J_{j+1}) \in \mathbf{sol}$, for each $i \in \{0, \ldots, m\}$ and each $j \in \{0, \ldots, n\}$ with the following properties: (i) $I_0 = J_0$, (ii) $\bigcup_{i=1}^m I_i = \bigcup_{j=1}^m J_j$ and (iii) $I_i \neq J_j$ for all $i \geq 1$ and all $j \geq 1$. Then the following two ratios are equal:

$$\sum_{i=1}^m \prod_{j=1}^i \frac{|I_j|}{|I_{j-1}|} = \frac{|\bigcup_{i=1}^m I_i|}{|I_0|} = \frac{|\bigcup_{i=1}^n J_j|}{|J_0|} = \sum_{i=1}^n \prod_{j=1}^i \frac{|J_j|}{|J_{j-1}|} .$$

We observe that the unstable spanning leaf segments $I_1, \ldots, I_m$ and $J_1, \ldots, J_n$ must be boundaries of Markov rectangles. Thus, a realized solenoid function σ_G must satisfy the following *boundary condition* for all such leaf segments:

$$\sum_{i=1}^m \prod_{j=1}^i \sigma_G\left(I_{j-1} : I_j\right) = \sum_{i=1}^n \prod_{j=1}^i \sigma_G\left(J_{j-1} : J_j\right) \quad . \tag{2.8}$$

Let K_1, K_2 and K_3 be the unstable spanning leaf segments as defined in Section 1.1. Let K_0 be the unstable spanning leaf segment in A such that $K_0 \cap K_1 = \{y_0\}$. Let the unstable spanning leaf segment I_1 be the right boundary of the Markov rectangle B and let the unstable spanning leaf segment I_2 be the right boundary of the Markov rectangle A (see Figure 4).

Lemma 2.3. *Let $\sigma_G : \mathbf{sol} \to \mathbb{R}^+$ be a realized solenoid function. The boundary condition holds for σ_G if the following conditions hold:*

$$\sigma_G(K_0 : K_1)(1 + \sigma_G(K_1 : K_2)) = \sigma_G(K_0 : I_1)(1 + \sigma_G(I_1 : I_2)) \tag{2.9}$$

and

$$\sigma_G(K_3 : K_2)(1 + \sigma_G(K_2 : K_1)) = \sigma_G(K_3 : I_2)(1 + \sigma_G(I_2 : I_1)). \tag{2.10}$$

Proof. Since I_1 and K_2 are the unstable boundaries of the Markov rectangle **B** and I_2 and K_1 are the unstable boundaries of the Markov rectangle **A**, then the boundary condition (2.8) corresponds to

$$\frac{|K_1|}{|K_0|}\left(1+\frac{|K_2|}{|K_1|}\right)=\frac{|K_1\cup K_2|}{|K_0|}=\frac{|I_1\cup I_2|}{|K_0|}=\frac{|I_1|}{|K_0|}\left(1+\frac{|I_2|}{|I_1|}\right)$$

and

$$\frac{|K_2|}{|K_3|}\left(1+\frac{|K_1|}{|K_2|}\right)=\frac{|K_1\cup K_2|}{|K_3|}=\frac{|I_1\cup I_2|}{|K_3|}=\frac{|I_2|}{|K_3|}\left(1+\frac{|I_1|}{|I_2|}\right).$$

Hence, the boundary condition for σ_G is given by (2.9) and (2.10). □

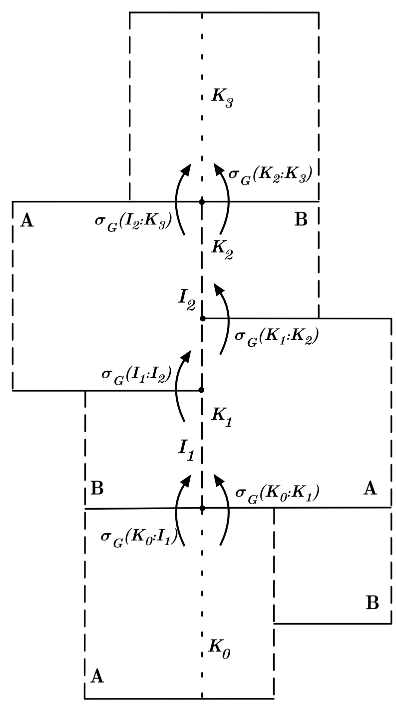


FIGURE 4. The boundary condition for the realized solenoid function σ_G.

2.9.1. *Solenoid functions.*

Definition 2.4. A function $\sigma : \mathbf{sol} \to \mathbb{R}^+$ is an *(unstable) solenoid function* if the following conditions hold: (i) σ is Hölder continuous; (ii) σ satisfies the matching condition given by the equalities (2.5), (2.6) and (2.7); and (iii) σ satisfies the boundary condition given by the equalities (2.9) and (2.10).

By Theorem 6.1 in [24], we have the following equivalence:

Lemma 2.5. *The map $r \to r|\mathbf{sol}$ gives a one-to-one correspondence between ratio functions and solenoid functions.*

Let $\mathcal{SOL}$ be the set consisting of all (unstable) solenoid functions. The set $\mathcal{SOL}$ has a natural metric. Combining Lemma 2.1 and Lemma 2.5, we obtain the following corollary.

Corollary 2.6. *The map $G \to r_G|\mathbf{sol}$ determines a one-to-one correspondence between C^{1+} conjugacy classes of Anosov diffeomorphisms G in $\mathcal{G}$ and solenoid functions $r_G|\mathbf{sol}$ in $\mathcal{SOL}$.*

3. Golden tilings

Recall from Section 1.1 the definitions of the unstable leaf segments $K_1, K_2, \ldots$, and the unstable leaf $W = \bigcup_{i \geq 1} K_i$. By construction, the set

$$\mathcal{L} = \{(K_i, K_{i+1}), \, i \in \mathbb{N}\}$$

is contained in **sol** and is dense in **sol**.

3.1. Realized golden sequences.

Lemma 3.1. *There is a well-defined map $G \to (a_i^G)_{i \in \mathbb{N}}$ that associates to each C^{1+} Anosov diffeomorphism G in $\mathcal{G}$ the sequence $(a_i^G)_{i \in \mathbb{N}}$ given by*

$$a_i^G \quad = \quad \lim_{n \to \infty} \frac{|G^{-n}(h_G(K_{i+1}))|}{|G^{-n}(h_G(K_i))|}.$$

Furthermore, $a_i^G = \sigma_G(K_i : K_{i+1})$.

Recall that h_G is the topological conjugacy between G and the Anosov automorphism G_A.

Proof. By Lemma 2.5 and equation (2.3) we get that $\sigma_G(K_i : K_{i+1}) = r_G^u(K_i : K_{i+1})$, where

$$r_G^u(K_i : K_{i+1}) \quad = \quad \lim_{n \to \infty} \frac{|G^{-n}(h_G(K_{i+1}))|}{|G^{-n}(h_G(K_i))|}$$

is well defined. Since, by construction, $a_i^G = r_G^u(K_i : K_{i+1})$, we get that a_i^G is well defined and $a_i^G = \sigma_G(K_i : K_{i+1})$. □

3.2. Fibonacci marking of the unstable leaf.

Lemma 3.2. *For every $i \in \mathbb{N}$ with Fibonacci decomposition $F_{n_0}, \ldots, F_{n_p}$ the following conditions hold:*

- *(i) $K_i \in \mathbf{B}$ and $K_{i+1} \in \mathbf{A}$ if either ($n_p = 1$ and n_{p-1} is odd) or ($n_p = 2$ and n_{p-1} is even);*
- *(ii) $K_i \in \mathbf{A}$ and $K_{i+1} \in \mathbf{B}$ if either ($n_p = 1$ and n_{p-1} is even) or ($n_p > 2$ and n_p is odd);*
- *(iii) $K_i, K_{i+1} \in \mathbf{A}$ if either ($n_p = 2$ and n_{p-1} is odd) or ($n_p > 2$ and n_p is even).*

Proof. Let $\pi : \mathbb{R}^2 \to \mathbb{T}$ be the natural projection, where $\mathbb{T} = \mathbb{R}^2/(v\mathbb{Z} \times w\mathbb{Z})$. Let $\mathbb{S} = \mathbb{R}/[1+\gamma]\mathbb{Z}$ be the clockwise oriented circle with the metric induced by the Euclidean metric on $\mathbb{R}$. Let $\pi_\mathbb{S} : \mathbb{R} \to \mathbb{S}$ be the natural projection. The projection $\pi_\mathbb{S}$ has the property that

$$\pi_\mathbb{S}(x) = \pi_\mathbb{S}(x + 1 + \gamma),$$

for every $x \in \mathbb{R}$. Let $i_\mathbb{S} : \mathbb{S} \to \mathbb{T}$ be the natural inclusion. The inclusion $i_\mathbb{S}$ has the property that

$$\pi(x, 0) = i_\mathbb{S} \circ \pi_\mathbb{S}(x),$$

for every $x \in \mathbb{R}$. Recall that K_0 is the unstable spanning leaf segment such that $K_0 \cap K_1 = \{y_0\}$, where $y_0 = \pi(0,0)$, and let $K_1, K_2, \ldots$ be the unstable spanning leaf segments such that $W = \bigcup_{i\geq 1} K_i$ (see Section 1.1). For every $i \in \mathbb{N}_0$, (i) let $y_i \in \mathbb{T}$ be the point given by $\{y_i\} = K_i \cap K_{i+1}$; (ii) let $z_i = i_{\mathbb{S}}^{-1}(y_i)$; and (iii) let $w_i \in [-1,\gamma]$ be such that $\pi_{\mathbb{S}}(w_i) = z_i$. Hence, for every $i \in \mathbb{N}_0$ (see Figure 5),

(i) if $w_i \in (-\gamma, 0)$, then $K_i \in \mathbf{A}$ and $K_{i+1} \in \mathbf{B}$;
(ii) if $w_i \in (0, \gamma^2)$, then $K_i, K_{i+1} \in \mathbf{A}$;
(iii) if $w_i \in [-1, -\gamma) \cup (\gamma^2, \gamma]$, then $K_i \in \mathbf{B}$ and $K_{i+1} \in \mathbf{A}$.

Let $g : \mathbb{S} \to \mathbb{S}$ be the golden rigid rotation with rotation number γ. The map g has the property that

$$g \circ \pi_{\mathbb{S}}(x) = \pi_{\mathbb{S}}(x + 1),$$

for every $x \in \mathbb{R}$. Since $G_A : \mathbb{T} \to \mathbb{T}$ is an Anosov automorphism, we obtain that $g(z_i) = z_{i+1}$, for every $i \in \mathbb{N}_0$. Let us denote by $\ell(y_0, y_i)$ the leaf segment with endpoints y_0 and y_i. Since $G_A : \mathbb{T} \to \mathbb{T}$ is the golden Anosov automorphism, if the leaf $\ell(y_0, y_i)$ contains m_A spanning leaf segments of the Markov rectangle A and m_B spanning leaf segments of the Markov rectangle B, then $G_A(\ell(y_0, y_i)) = \ell(y_0, G_A(y_i))$ contains $m_A + m_B$ spanning leaf segments of the Markov rectangle A and m_A spanning leaf segments of the Markov rectangle B. Hence, by induction, we have that $G_A(y_{F_i}) = y_{F_{i+1}}$, where $F_1, F_2, \ldots$ are the Fibonacci numbers. Thus, for every $i \in \mathbb{N}$, we have that $G_A^{i-1}(y_1) = y_{F_i}$, and so $d(y_0, y_{F_i}) = \gamma^i$ and $\pi((-\gamma)^i, 0) = y_{F_i}$. Thus $g^{F_i}(z_0) = z_{F_i} = \pi_{\mathbb{S}}((-\gamma)^i)$. Since g is the golden rigid rotation, we have that

$$g^{F_i}(\pi_{\mathbb{S}}(x)) = \pi_{\mathbb{S}}(x + (-\gamma)^i), \tag{3.1}$$

for every $x \in \mathbb{R}$ and $i \in \mathbb{N}$. Hence, for every $i \in \mathbb{N}$ with Fibonacci decomposition $F_{n_0}, \ldots, F_{n_p}$, we obtain

$$z_i = g^i(z_0) = g^{F_{n_0}+\cdots+F_{n_p}}(z_0).$$

Thus, by equality (3.1), we have that

$$g^{F_{n_0}+\cdots+F_{n_p}}(z_0) = \pi_{\mathbb{S}}\left(\sum_{i=0}^{p}(-\gamma)^{n_i}\right).$$

Noting that $\sum_{i=0}^{+\infty} \gamma^{2i} = (1-\gamma^2)^{-1} = \gamma^{-1}$, we obtain

$$\sum_{i=0}^{p}(-\gamma)^{n_i} < \sum_{j\geq 0} \gamma^{2+2j} = \gamma$$

and

$$\sum_{i=0}^{p}(-\gamma)^{n_i} > \sum_{j\geq 0} -\gamma^{1+2j} = -1.$$

Therefore, taking $w_i = \sum_{j=0}^{p}(-\gamma)^{n_j} \in [-1,\gamma]$ we obtain that $\pi_{\mathbb{S}}(w_i) = z_i$. Now, there are six distinct cases to consider depending upon the Fibonacci decomposition $F_{n_0}, \ldots, F_{n_p}$ of i (see Figure 6):

(i) if $n_p = 1$ and n_{p-1} is odd, then $w_i \in [-1, -\gamma)$, and so $K_i \in \mathbf{B}$ and $K_{i+1} \in \mathbf{A}$;
(ii) if $n_p = 1$ and n_{p-1} is even, then $w_i \in (-\gamma, -\gamma^2)$, and so $K_i \in \mathbf{A}$ and $K_{i+1} \in \mathbf{B}$;
(iii) if $n_p = 2$ and n_{p-1} is odd, then $w_i \in (\gamma^3, \gamma^2)$, and so $K_i, K_{i+1} \in \mathbf{A}$;
(iv) if $n_p = 2$ and n_{p-1} is even, then $w_i \in (\gamma^2, \gamma]$, and so $K_i \in \mathbf{B}$ and $K_{i+1} \in \mathbf{A}$;

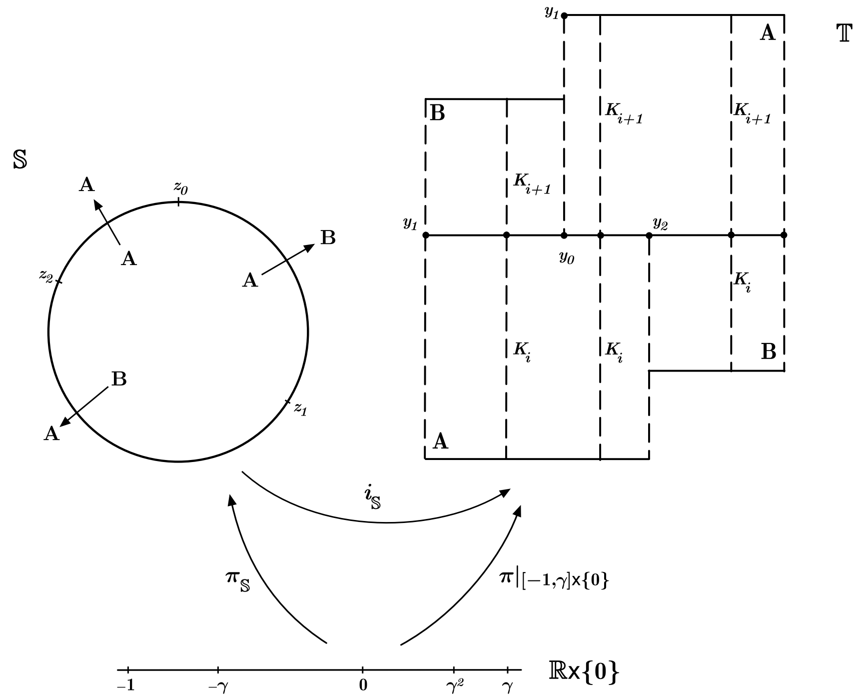


FIGURE 5. The map $i_{\mathbb{S}} \circ \pi_{\mathbb{S}}$.

(v) if $n_p > 2$ and n_p is odd, then $w_i \in (-\gamma^2, 0)$, and so $K_i \in \mathbf{A}$ and $K_{i+1} \in \mathbf{B}$;
(vi) if $n_p > 2$ and n_p is even, then $w_i \in (0, \gamma^3)$, and so $K_i, K_{i+1} \in \mathbf{A}$. □

3.3. **Fibonacci shift.** We define the *Fibonacci shift* $\sigma : \mathbb{N} \to \mathbb{N}$ as follows. For every $i \in \mathbb{N}$, let $F_{n_0}, \ldots, F_{n_p}$ be the Fibonacci decomposition associated to i, i.e. $i = F_{n_0} + \cdots + F_{n_p}$. If $n_p \neq 1$, then we define $\sigma(i) = F_{n_0+1} + \cdots + F_{n_p+1}$. If $n_p = 1$ and n_{p-1} is odd, then we define $\sigma(i) = F_{n_0+1} + \cdots + F_{n_{p-1}+1} + F_1$. If $n_p = 1$ and n_{p-1} is even, then we define $\sigma(i) = F_{n_0+1} + \cdots + F_{n_{p-1}+1} + F_2$. Hence, the inverse of the Fibonacci shift $\sigma^{-1}(i)$ is given as follows: if $n_p \neq 1$, then $\sigma^{-1}(i) = F_{n_0-1} + \cdots + F_{n_p-1}$; if $n_p = 1$ and n_{p-1} is even, then $\sigma^{-1}(i) = F_{n_0-1} + \cdots + F_{n_{p-1}-1} + F_1$; if $n_p = 1$ and n_{p-1} is odd, then $\sigma^{-1}(i) = \emptyset$.

Remark 3.3. We observe that for $F_{n_p} = F_1$ the definition of the Fibonacci shift is somewhat unnatural. This is due to the fact that we consider, for simplicity, the Fibonacci sequence $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$ instead of the sequence $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$. If we consider the sequence $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$, then we have to change the Fibonacci decomposition of the number i accordingly with the following rule: if n_{p-1} is odd and $i - (F_{n_0} + \cdots + F_{n_{p-1}}) = 1$, then $n_p = 0$. In this case, for every $i \in \mathbb{N}$, we define $\sigma(i) = F_{n_0+1} + \cdots + F_{n_{p-1}+1} + F_{n_p+1}$. We get that if $n_p \neq 0$, then $\sigma^{-1}(i) = F_{n_0-1} + \cdots + F_{n_p-1}$, and if $n_p = 0$, then $\sigma^{-1}(i) = \emptyset$. We claim that for this new Fibonacci decomposition all the statements in the paper will hold with the corresponding simple alterations.

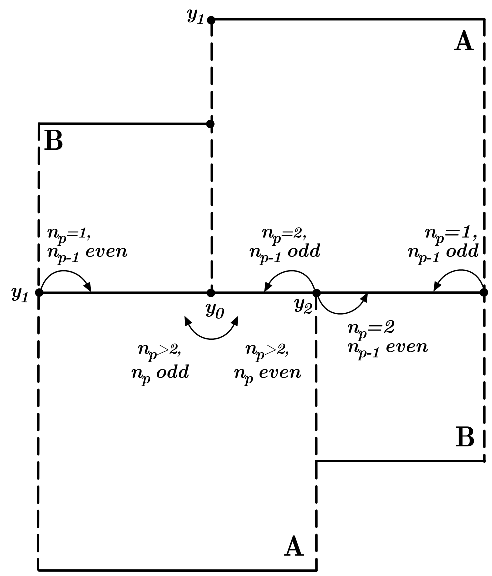


FIGURE 6. The location of the point y_i depending upon the Fibonacci decomposition of i.

3.4. **Matching condition.** We say that a sequence $(a_i)_{i\in\mathbb{N}}$ satisfies the *matching condition* if, for every $i = F_{n_0} + \cdots + F_{n_p}$, the following conditions hold:

(i) If either ($n_p = 1$ and n_{p-1} is odd) or ($n_p = 2$ and n_{p-1} is even), then

$$a_{\sigma(i)} = a_i \left(a_{\sigma(i)+1} + 1\right)^{-1}.$$

(ii) If either ($n_p = 1$ and n_{p-1} is even) or ($n_p > 2$ and n_p is odd), then

$$a_{\sigma(i)} = a_i \left(a^{-1}_{\sigma(i)-1} + 1\right).$$

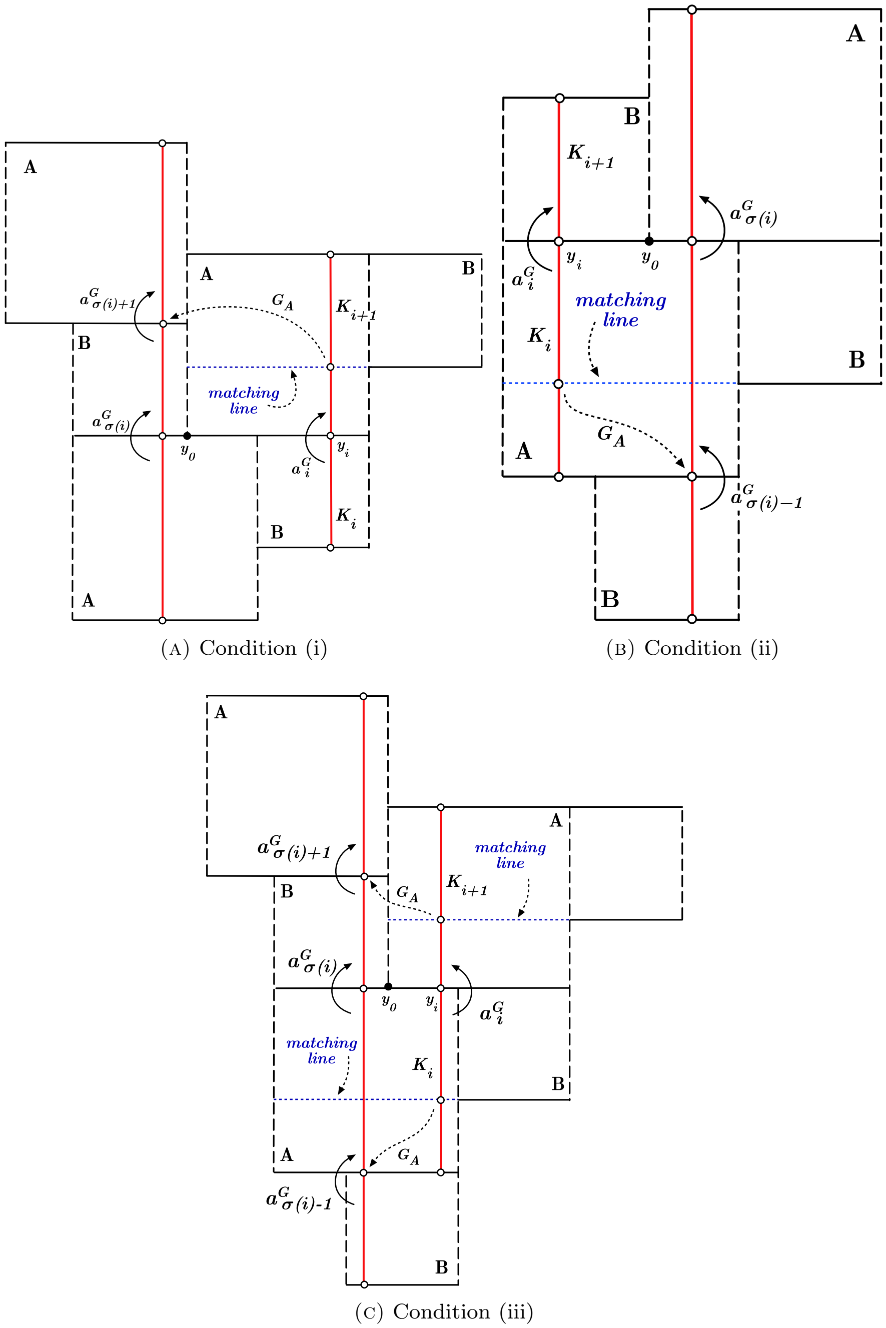
(iii) If either ($n_p = 2$ and n_{p-1} odd) or ($n_p > 2$ and n_p is even), then

$$a_{\sigma(i)} = \frac{a_i \left(1 + a_{\sigma(i)-1}\right)}{a_{\sigma(i)-1}\left(1 + a_{\sigma(i)+1}\right)} .$$

Lemma 3.4. *The sequence $(a_i^G)_{i\in\mathbb{N}}$ satisfies the matching condition.*

Proof. By Lemma 3.1, we have that $a_i^G = \sigma_G(K_i : K_{i+1})$, for every $i \in \mathbb{N}$. Hence Lemma 3.4 follows from putting together Lemma 2.2 and Lemma 3.2. □

Remark 3.5. Every sequence $(b_i)_{i\in\mathbb{N}\setminus\sigma(\mathbb{N})}$ determines, uniquely, a sequence $(a_i)_{i\in\mathbb{N}}$ satisfying the matching condition as follows: for every $i \in \mathbb{N}\setminus\sigma(\mathbb{N})$, define $a_i = b_i$ and, for every $i \in \sigma(\mathbb{N})$, define $a_{\sigma(i)}$ using the matching condition and the elements



(A) Condition (i)

(B) Condition (ii)

(C) Condition (iii)

FIGURE 7. The matching condition.

a_j of the sequence with

$$j \in \{j : 2 \leq j < \sigma(i) \vee j \in \mathbb{N} \backslash \sigma(\mathbb{N})\}.$$

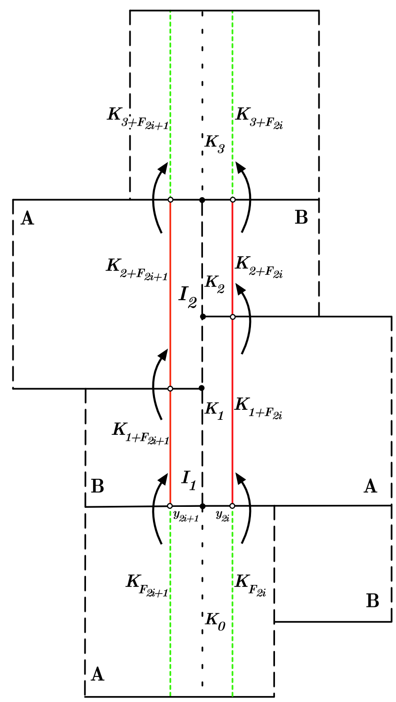


FIGURE 8. The boundary condition.

3.5. **Boundary condition.** A sequence $(a_i)_{i\in\mathbb{N}}$ satisfies the *boundary condition* if the following limits are well defined and satisfy the inequalities:

(i) $\lim_{i\to+\infty} a^{-1}_{F_i+2}\left(1+a^{-1}_{F_i+1}\right) \neq 0.$

(ii) $\lim_{i\to+\infty} a_{F_i}\left(1+a_{F_i+1}\right) \neq 0.$

Lemma 3.6. *The sequence $(a^G_i)_{i\in\mathbb{N}}$ satisfies the boundary condition.*

Proof. We observe that $d(K_{F_n}, K_0) = \gamma^n$, $d(K_{F_{2n+1}+1}, I_1) = \gamma^{2n+1}$, $d(K_{F_{2n+1}+2}, I_2) = \gamma^{2n+1}$, $d(K_{F_{2n}+1}, K_1) = \gamma^{2n}$, $d(K_{F_{2n}+2}, K_2) = \gamma^{2n}$ and $d(K_{F_n+3}, K_3) = \gamma^n$ (see Figure 8). By continuity of σ_a, we have that

$$\begin{aligned}
&\lim_{n\to\infty} a^G_{F_{2n}}(1 + a^G_{F_{2n}+1})\\
&\qquad = \lim_{n\to\infty} \sigma_G(K_{F_{2n}} : K_{F_{2n}+1})(1 + \sigma_G(K_{F_{2n}+1} : K_{F_{2n}+2}))\\
&\qquad = \sigma_G(K_0 : K_1)(1 + \sigma_G(K_1 : K_2))
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} a^G_{F_{2n+1}}(1 + a^G_{F_{2n+1}+1})\\
&\qquad = \lim_{n\to\infty} \sigma_G(K_{F_{2n+1}} : K_{F_{2n+1}+1})(1 + \sigma_G(K_{F_{2n+1}+1} : K_{F_{2n+1}+2}))\\
&\qquad = \sigma_G(K_0 : I_1)(1 + \sigma_G(I_1 : I_2)).
\end{aligned}$$

Hence, by equality (2.9), we obtain that the golden sequence $(a_i^G)_{i\in\mathbb{N}}$ satisfies the boundary condition (i). By continuity of σ_G, we have that

$$\begin{aligned}\lim_{n\to\infty}&(a^G_{F_{2n}+2})^{-1}(1+(a^G_{F_{2n}+1})^{-1})\\ &=\lim_{n\to\infty}\sigma_G(K_{F_{2n}+3}:K_{F_{2n}+2})(1+\sigma_G(K_{F_{2n}+2}:K_{F_{2n}+1}))\\ &=\sigma_G(K_3:K_2)(1+\sigma_G(K_2:K_1))\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}&(a^G_{F_{2n+1}+2})^{-1}(1+(a^G_{F_{2n+1}+1})^{-1})\\ &=\lim_{n\to\infty}\sigma_G(K_{F_{2n+1}+3}:K_{F_{2n+1}+2})(1+\sigma_G(K_{F_{2n+1}+2}:K_{F_{2n+1}+1}))\\ &=\sigma_G(K_3:I_2)(1+\sigma_G(I_2:I_1)).\end{aligned}$$

Hence, by equality (2.10) we obtain that the golden sequence $(a_i^G)_{i\in\mathbb{N}}$ satisfies the boundary condition (ii). □

3.6. **The exponentially fast Fibonacci repetitive property.** A sequence $(a_i)_{i\in\mathbb{N}}$ is said to be *exponentially fast Fibonacci repetitive* if there exist constants $C\ge 0$ and $0<\mu<1$ such that

$$|a_{i+F_m}-a_i|\le C\mu^m,$$

for every $m\ge 5$ and $3\le i<F_{m-1}$ and, also, for $i\in\{1,2\}$ if m is even.

Lemma 3.7. *The sequence (a_i^G) satisfies the exponentially fast Fibonacci repetitive property.*

Proof. For every $m\ge 5$, we have that either $m=2n$ or $m=2n+1$ for some $n\ge 2$ (see Figure 9). Recall that $K_i\cap K_{i+1}=\{y_i\}$, for every $i\in\mathbb{N}_0$.

(i) Case $m=2n$. For $1\le i<F_{2n-1}$, the unstable spanning leaf segments K_i, K_{i+1}, $K_{i+F_{2n}}$ and $K_{i+1+F_{2n}}$ belong to $G^{2n}(\mathbf{A})$. Hence, we obtain that

$$d(K_i,K_{i+F_{2n}})\le C_0|y_i-y_{i+F_{2n}}|\le C_0\gamma^{2n},$$

for some $C_0\ge 1$ and $0<\gamma<1$. By Hölder continuity of the solenoid function, there exist constants $C\ge 1$ and $\alpha<1$ such that

$$\begin{aligned}|a_i^G-a^G_{i+F_{2n}}| &= |\sigma_G(K_i:K_{i+1})-\sigma_G(K_{i+F_{2n}}:K_{i+1+F_{2n}})|\\ &< C\left(\gamma^{2n}\right)^{\alpha}\\ &= C\left(\gamma^{\alpha}\right)^{2n}.\end{aligned}$$

(ii) Case $m=2n+1$. For $3\le i<F_{2n}$, the unstable spanning leaf segments K_i, K_{i+1}, $K_{i+F_{2n+1}}$ and $K_{i+1+F_{2n+1}}$ belong to $G^{2n}(\mathbf{B})$. Hence, we obtain that

$$d(K_i,K_{i+F_{2n+1}})\le C_0|y_i-y_{i+F_{2n+1}}|\le C_0\gamma^{2n+1},$$

for some $C_0\ge 1$ and $0<\gamma<1$. By Hölder continuity of the solenoid function, there exist constants $C\ge 1$ and $\alpha<1$ such that

$$\begin{aligned}|a_i^G-a^G_{i+F_{2n+1}}| &= |\sigma_G(K_i:K_{i+1})-\sigma_G(K_{i+F_{2n+1}}:K_{i+1+F_{2n+1}})|\\ &< C\left(\gamma^{2n+1}\right)^{\alpha}\\ &= C\gamma^{\alpha}\left(\gamma^{\alpha}\right)^{2n}.\end{aligned}$$

Hence, the sequence (a_i^G) satisfies the exponentially fast Fibonacci repetitive property with $\mu=\gamma^\alpha$. □

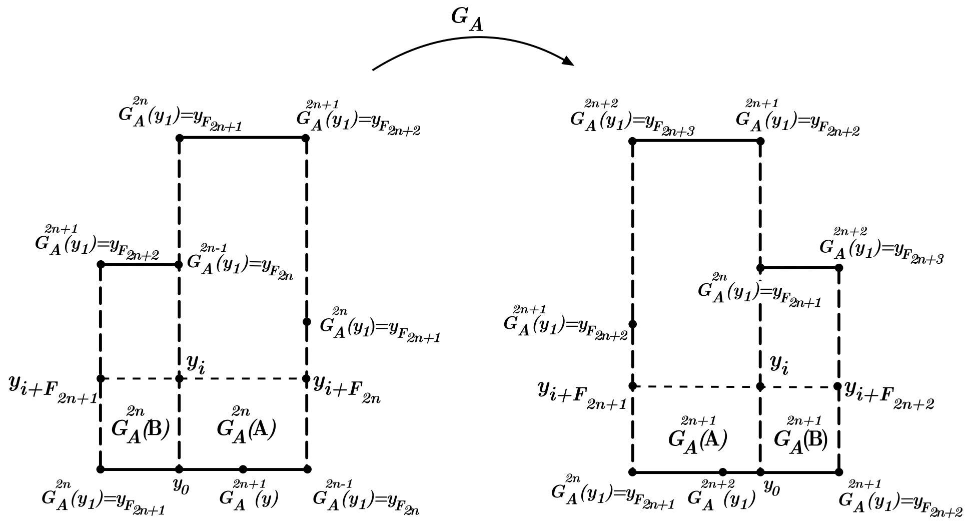


FIGURE 9. The exponentially fast Fibonacci repetitive condition.

3.7. **Golden tilings.** A *tiling* $\mathcal{T} = \{I_i \subset \mathbb{R} : i \in \mathbb{N}\}$ of the positive real line is a collection of tiling intervals I_i, with the following properties:

(i) the tiling intervals are closed intervals;
(ii) any two distinct intervals have disjoint interiors;
(iii) the union $\bigcup_{i\in\mathbb{N}} I_i$ is equal to the positive real line;
(iv) for every $i \in \mathbb{N}$ the intersection of the tiling intervals I_i and I_{i+1} is only a point, which is an endpoint, simultaneously, of both intervals.

The tilings $\mathcal{T}_1 = \{I_i \subset \mathbb{R} : i \in \mathbb{N}\}$ and $\mathcal{T}_2 = \{J_i \subset \mathbb{R} : i \in \mathbb{N}\}$ of the positive real line are in the same affine class if there exists an affine map $h : \mathbb{R} \to \mathbb{R}$ such that $h(I_i) = J_i$, for every $i \in \mathbb{N}$. Thus, every positive sequence $(a_i)_{i\in\mathbb{N}}$ determines a unique affine class of tilings $\mathcal{T} = \{I_i \subset \mathbb{R} : i \in \mathbb{N}\}$ such that $a_i = |I_{i+1}| / |I_i|$, and vice versa.

Definition 3.8. A *golden sequence* $(a_i)_{i\in\mathbb{N}}$ is an exponentially fast Fibonacci repetitive sequence that satisfies the matching and the boundary conditions. A tiling $\mathcal{T} = \{I_i \subset \mathbb{R} : i \in \mathbb{N}\}$ of the positive real line is *golden* if the corresponding sequence $(a_i = |I_{i+1}|/|I_i|)_{i\in\mathbb{N}}$ is a golden sequence.

We say that a golden tiling $\mathcal{T}_R$ is *rigid* if its associated golden sequence is rigid (see Definition 1.2).

3.8. **Proof of Theorem 1.1.** By Lemma 3.1, the map $G \to (a_i^G)_{i\in\mathbb{N}}$ determines a correspondence between Anosov diffeomorphisms G in $\mathcal{G}$ and golden sequences such that $a_i^G = \sigma_G(K_i : K_{i+1})$. Putting together Lemma 3.4, Lemma 3.6 and Lemma 3.7, we get that $(a_i^G)_{i\in\mathbb{N}}$ is a golden sequence. By Corollary 2.6, any two C^{1+} Anosov diffeomorphisms, G_1 and G_2, that are C^{1+} smooth conjugate determine the same solenoid functions $\sigma_{G_1} = \sigma_{G_2}$. Hence, by Lemma 3.1, $(a_i^{G_1})_{i\in\mathbb{N}} = (a_i^{G_2})_{i\in\mathbb{N}}$.

Conversely, given a golden sequence $(a_i)_{i\in\mathbb{N}}$ we construct a solenoid function σ_a in **sol** as follows. Recall that $\mathbb{L} = \{(K_i : K_{i+1}), i \in \mathbb{N}\}$ is a dense set in **sol**. We define $\sigma_a(K_i : K_{i+1}) = a_i$, for every $(K_i : K_{i+1}) \in \mathbb{L}$. Since the sequence

$(a_i)_{i\in\mathbb{N}}$ is exponentially fast Fibonacci repetitive, similar to the proof of Lemma 3.7, we get that $\sigma_a|\mathbb{L}$ is Hölder continuous. Hence, using the fact that $\mathbb{L}$ is dense in **sol**, we define σ_a in **sol** as the unique Hölder continuous extension of $\sigma_a|\mathbb{L}$ to **sol**. Now, it is enough to check that the Hölder continuous function σ_a in **sol** satisfies the matching and the boundary condition. Since $(a_i)_{i\in\mathbb{N}}$ satisfies the golden matching condition, similar to the proof of Lemma 3.4, we have that $\sigma_a|\mathbb{L}$ satisfies the matching condition in $\mathbb{L}$. Hence, using the fact that σ_a in **sol** is a continuous function, we get that the σ_a in **sol** also satisfies the matching condition. Recall, from Section 1.1, the definition of K_0, K_1, K_2 and K_3. Recall that the spanning leaf segments I_1 and I_2 are, respectively, the right boundaries of the Markov rectangles B and A, as in Section 2.9. We observe that $d(K_{F_n}, K_0) = \gamma^n$, $d(K_{F_{2n+1}+1}, I_1) = \gamma^{2n+1}$, $d(K_{F_{2n+1}+2}, I_2) = \gamma^{2n+1}$, $d(K_{F_{2n}+1}, K_1) = \gamma^{2n}$, $d(K_{F_{2n}+2}, K_2) = \gamma^{2n}$ and $d(K_{F_n+3}, K_3) = \gamma^n$. By continuity of σ_a, we have that

$$\begin{aligned}\sigma_a(K_0 : K_1)&(1+\sigma_a(K_1 : K_2))\\ &= \lim_{n\to\infty} \sigma_a(K_{F_{2n}} : K_{F_{2n}+1})(1+\sigma_a(K_{F_{2n}+1} : K_{F_{2n}+2}))\\ &= \lim_{n\to\infty} a_{F_{2n}}(1+a_{F_{2n}+1})\end{aligned}$$

and

$$\begin{aligned}\sigma_a(K_0 : I_1)&(1+\sigma_a(I_1 : I_2))\\ &= \lim_{n\to\infty} \sigma_a(K_{F_{2n+1}} : K_{F_{2n+1}+1})(1+\sigma_a(K_{F_{2n+1}+1} : K_{F_{2n+1}+2}))\\ &= \lim_{n\to\infty} a_{F_{2n+1}}(1+a_{F_{2n+1}+1}).\end{aligned}$$

Hence, by the boundary condition (i) of the golden sequence $(a_i)_{i\in\mathbb{N}}$, we obtain that σ_a satisfies equality (2.9). By continuity of σ_a, we have that

$$\begin{aligned}\sigma_a(K_3 : K_2)&(1+\sigma_a(K_2 : K_1))\\ &= \lim_{n\to\infty} \sigma_a(K_{F_{2n}+3} : K_{F_{2n}+2})(1+\sigma_a(K_{F_{2n}+2} : K_{F_{2n}+1}))\\ &= \lim_{n\to\infty} a^{-1}_{F_{2n}+2}(1+a^{-1}_{F_{2n}+1})\end{aligned}$$

and

$$\begin{aligned}\sigma_a(K_3 : I_2)&(1+\sigma_a(I_2 : I_1))\\ &= \lim_{n\to\infty} \sigma_a(K_{F_{2n+1}+3} : K_{F_{2n+1}+2})(1+\sigma_a(K_{F_{2n+1}+2} : K_{F_{2n+1}+1}))\\ &= \lim_{n\to\infty} a^{-1}_{F_{2n+1}+2}(1+a^{-1}_{F_{2n+1}+1}).\end{aligned}$$

Hence, by the boundary condition (ii) of the golden sequence $(a_i)_{i\in\mathbb{N}}$, we obtain that σ_a satisfies equality (2.10). Therefore, σ_a is a solenoid function. □

4. Complete set of holonomies

Let $h_G : \mathbb{T} \to \mathbb{T}$ be the topological conjugacy between the Anosov automorphism G_A and G. The rectangles $h_G(\mathbf{A})$ and $h_G(\mathbf{B})$ form a Markov partition $\mathcal{M}_G$ of G. Suppose that M and N are Markov rectangles and that $x \in int(M)$ and $y \in int(N)$. We say that x and y are *stable holonomically related* if (i) there is an stable leaf segment $\ell^u(x, y)$ such that $\partial \ell^u(x, y) = \{x, y\}$, and (ii) $\ell^u(x, y) \subset \ell^u(x, M) \cup \ell^u(y, N)$. Let $P = P_{\mathcal{M}}$ be the set of all pairs (M, N) such that there are points $x \in int(M)$ and $y \in int(N)$ unstable holonomically related.

For every Markov rectangle $M \in \mathcal{M}_G$, choose an unstable spanning leaf segment $\ell(x, M)$ in M for some $x \in M$. Let $\mathcal{I} = \{\ell_M : M \in \mathcal{M}\}$. For every pair $(M, N) \in P$, there are maximal leaf segments $\ell^D_{(M,N)} \subset \ell_M$, $\ell^C_{(M,N)} \subset \ell_N$ such that the unstable holonomy $h_{(M,N)} : \ell^D_{(M,N)} \to \ell^C_{(M,N)}$ is well defined. We call such holonomies $h_{(M,N)} : \ell^D_{(M,N)} \to \ell^C_{(M,N)}$ the *unstable primitive holonomies* associated to the Markov partition $\mathcal{M}_G$. The *complete set of unstable holonomies* $\mathcal{H}_G$ consists of all stable primitive holonomies and their inverses. In Figure 10, we exhibit the complete set of unstable holonomies

$$\mathcal{H}_G = \left\{ h_{(\mathbf{A},\mathbf{A})}, h^{-1}_{(\mathbf{A},\mathbf{A})}, h_{(\mathbf{A},\mathbf{B})}, h^{-1}_{(\mathbf{A},\mathbf{B})}, h_{(\mathbf{B},\mathbf{A})}, h^{-1}_{(\mathbf{B},\mathbf{A})} \right\}$$

associated to the Markov partition $\mathcal{M}_G$.

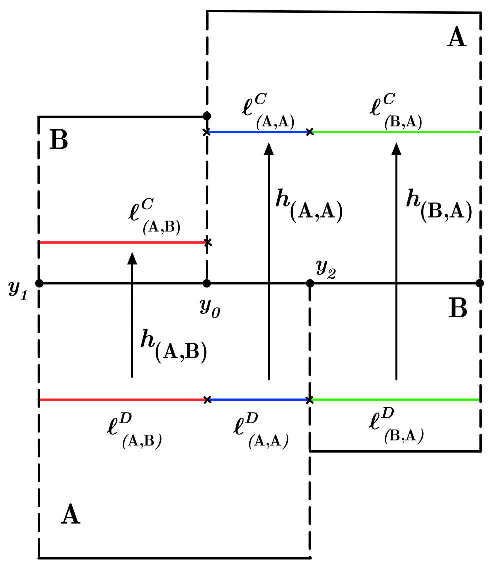


FIGURE 10. A complete set of unstable holonomies $\mathcal{H}_G$ associated to the Markov partition $\mathcal{M}_G$.

A diffeomorphism $\theta : I \to J$ is said to be $C^{1+zygmund}$ if θ is C^1 and the derivative θ' satisfies the zygmund condition, i.e. for all points $x, y \in I$,

$$\left| \theta'(x) + \theta'(y) - 2\theta' \left(\frac{x+y}{2} \right) \right| = \chi_\theta \left(|y - x| \right),$$

where the function χ_θ is such that $\chi_\theta(t) \to 0$ when $t \to 0$. In particular, a $C^{2+\beta}$ diffeomorphism, with $\beta > 0$, is a $C^{1+zygmund}$ diffeomorphism. The importance of this smooth class follows from the fact that it corresponds to maps that distort cross-ratios of quadruples of points in I by an amount that is $o(|I|)$ (see [7], [18] and [28]).

Definition 4.1. A complete set of unstable holonomies $\mathcal{H}_G$ is $C^{1+zygmund}$ if all holonomies in $\mathcal{H}_G$ are $C^{1+zygmund}$ with respect to the atlas $\mathcal{L}^s(G, \rho)$.

4.1. **Proof of Theorem 1.3.** By Lemma 3.3 and Theorem 4.1 in [22], if G has a $C^{1+zygmund}$ complete system of unstable holonomies, then $\sigma_G = \sigma_{G_A}$. By Lemma 3.2, for $i \in \mathbb{N}$ with Fibonacci decomposition $F_{n_0}, \ldots, F_{n_p}$, we have that the following conditions hold:

(i) If either ($n_p = 1$ and n_{p-1} is odd) or ($n_p = 2$ and n_{p-1} is even), then $K_i \in \mathbf{B}$ and $K_{i+1} \in \mathbf{A}$. Hence, $a_i = \sigma_{G_A}(K_i : K_{i+1}) = \gamma^{-1}$.

(ii) If either ($n_p = 1$ and n_{p-1} is even) or ($n_p > 2$ and n_p is odd), then $K_i \in \mathbf{A}$ and $K_{i+1} \in \mathbf{B}$. Hence, $a_i = \sigma_{G_A}(K_i : K_{i+1}) = \gamma$.

(iii) If either ($n_p = 2$ and n_{p-1} is odd) or ($n_p > 2$ and n_p is even), then $K_i, K_{i+1} \in \mathbf{A}$. Hence, $a_i = \sigma_{G_A}(K_i : K_{i+1}) = 1$.

Thus, from conditions (i), (ii) and (iii) we conclude that $(a_i^G)_{i \in \mathbb{N}}$ is the rigid golden sequence. □

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