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Skew-Product Maps and Heterodimensional Cycles

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Ao meu marido, José.

Ao meu filho, José Eduardo.

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Resumo

Nesta tese estudam-se as dinâmicas geradas pela criação de ciclos heterodimensionais, seja do tipo parcialmente hiperbólicas com folheações invariantes e dinâmica central unidimensional, seja associada a produtos torcidos.

Num primeiro cenário, considera-se uma família, a um parâmetro, de difeomorfismos exibindo um desdobramento de um ciclo heterodimensional associado a duas selas com diferentes índices e cuja dinâmica central é dada por um difeomorfismo côncavo. O estudo da dinâmica semi-local desta família, depois do desdobramento do ciclo, é então reduzido à análise de um sistema iterado de funções, obtido pela composição de potências da aplicação côncava com uma translação.

Motivado pelo estudo deste tipo de sistemas iterados de funções, introduz-se um modelo mais geral de sistemas parcialmente hiperbólicos: os produtos torcidos associados à aplicação *shift de Bernoulli* de n-símbolos.

Em ambos os casos, obtêm-se condições que garantem a prevalência de hiperbolicidade ou, em sentido contrário, a prevalência de não hiperbolicidade.

No caso dos produtos torcidos e assumindo *hipóteses de não hiperbolicidade*, prova-se a existência de uma medida invariante, ergódica e não-hiperbólica com um suporte não trivial. Encontra-se ainda um limite superior para o crescimento do número de órbitas periódicas.

Introduz-se ainda uma família modelo de difeomorfismos, a dois parâmetros, em que um dos parâmetros está relacionado com o desdobramento do ciclo heterodimensional do tipo descrito acima, e o outro associado a uma função côncava especial que fornece a dinâmica central. Neste caso é possível localizar, em função dos dois parâmetros, intervalos escalonados de hiperbolicidade e de não hiperbolicidade e em simultâneo descrever as bifurcações secundárias associadas à transição das regiões de hiperbolicidade para as de não hiperbolicidade.

Abstract

In this thesis we study the dynamics generated by the creation of heterodimensional cycles, either of the partially hyperbolic type, with invariant foliations and one-dimensional central dynamics, or associated to skew-product maps.

In the first scenario, we consider a one-parameter family unfolding a heterodimensional cycle associated to two saddles of different indices and such that the central dynamics is given by a concave diffeomorphism. The study of the semi-local dynamics of this family, after the unfolding of the cycle, is then reduced to the analysis of a system of iterated functions, obtained by compositions of powers of the concave map with a translation.

Motivated by the study of the this kind of iterated systems of functions, we introduce a more general model for partially hyperbolic systems: the skew-product maps associated to the *bernoulli shift* of n-symbols.

In both cases we obtain conditions which ensure prevalence of hyperbolicity or, in the opposite direction, prevalence of non-hyperbolicity.

In the skew-products case and under some *non-hyperbolicity hypothesis*, we prove the existence of an invariant ergodic and non-hyperbolic measure with an uncountable support. We also obtain an upper bound for the growth of the number of periodic orbits.

We also introduce a two-parameter family model of diffeomorphisms, being one the parameters associated to the unfolding of a heterodimensional cycle of the type described above, and the other associated to a special concave function that gives the central dynamics. In this case, depending on the two parameters, we are able to identify scalled intervals of hyperbolicity and of non-hyperbolicity, and furthermore describe the secondary bifurcations associated to the transition from hyperbolicity to non-hyperbolicity.

Contents

Introduction

Bifurcation theory studies transitions from hyperbolic (stable) to non-hyperbolic (unstable) regimes. There are two main sort of bifurcations: loss of hyperbolicity of periodic orbits (saddle node, flip, Hopf bifurcation) and generation of cycles. In this work we study bifurcations via creation of cycles and how these bifurcations are related to the loss of hyperbolicity of periodic orbits.

There are two types of cycles: equidimensional cycles involving only saddles with the same index (dimension of the unstable manifold) and heterodimensional cycles involving periodic points of different indices. The equidimensional cycles where well studied since the seventies (see [PT87]) and are associated to homoclinic tangencies. In this thesis the main emphasis is the study of bifurcations through heterodimensional cycles.

Let M be a compact, connected and boundaryless n-dimensional Riemannian manifold and $f : M \to M$ a diffeomorphism having a pair of hyperbolic periodic points P and Q with different indices. The diffeomorphism f exhibits a heterodimensional cycle associated to P and Q if the stable manifold $W^s(P, f)$ intersects the unstable manifold $W^u(Q, f)$ of Q, and the unstable manifold $W^u(P, f)$ of P intersects the stable manifold $W^s(Q, f)$ of Q. In this thesis, we consider the codimension one-case, that is, the index of Q, $\text{index}(Q)$, is equal to $\text{index}(P) + 1$.

Typically, heterodimensional cycles generate two transitive sets containing saddles of different indices. These sets are homoclinic classes of periodic points. The homoclinic class of a (hyperbolic) saddle P of a diffeomorphism f, denoted by $H(P, f)$, is the closure of the transverse intersections of the orbits of the stable and unstable manifolds of P. Two saddles P and Q are homoclinically related if the stable manifold of P, $W^s(P, f)$, intersects transversally the unstable manifold of Q, $W^u(Q, f)$, and the unstable manifold of P intersects transversally the stable manifold of Q. We

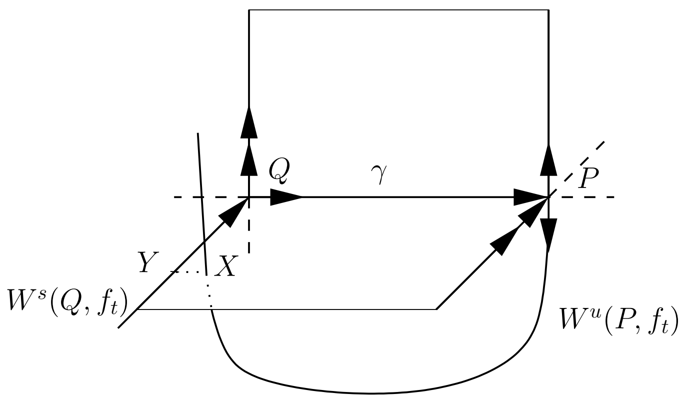

Figure 1: A heterodimensional cycle

observe that two saddles homoclinically related have the same index and their homoclinic classes coincide. In fact, we can define the homoclinic class of a saddle P as the closure of the saddles homoclinically related to P. Recall that if a homoclinic class contain saddles of different indices, then it is not hyperbolic.

We consider a family of diffeomorphisms $(f_t)_{t\in[-1,1]}$, with $f_0 = f$, unfolding a heterodimensional cycle at $t = 0$, associated to the periodic saddles P and Q, and we assume that the arc has a *first bifurcation* at $t = 0$, that is, f_t satisfies Axiom A and the no-cycles condition for every $t < 0$. The goal is to describe the dynamics of the diffeomorphisms f_t in $(0, \epsilon)$ for a "large set of parameters" (a set of nonzero Lebesgue measure with positive relative density at the bifurcation $t = 0$).

We assume that the cycle associated to P and Q is "generic", that is,

- $W^s(P, f) \cap W^u(Q, f)$ is non-empty and transverse,
- the intersection $W^s(Q, f) \cap W^u(P, f)$ is quasi-transverse, i.e.

$$\dim(T_X W^u(P, f) + T_X W^s(Q, f)) = n - 1,$$

  for all $X \in W^s(Q, f) \cap W^u(P, f)$.

Following the approach in [PT93] the idea is to study the dynamics in a *neighborhood of the cycle*, that is, an open set $\mathcal{W}$ containing all the elements of the cycle: the periodic points Q and P, the intersections $W^s(P, f) \cap W^u(Q, f)$ and $W^u(P, f) \cap W^s(Q, f)$ between their invariant manifolds. The objective is to describe the *resulting non-wandering set*,

$$\Omega(f_t)' = \Omega(f_t) \cap \mathcal{W}, \quad \text{for small } t > 0,$$

where $\Omega(f)$ is the set of non-wandering points, associated to the unfolding of the cycle, that is, to characterize the dynamics of $\Omega(f_t)'$.

Let $t > 0$ small. We omit the dependence of the periodic points on the parameter t. We say that the homoclinic classes of P and Q are *intermingled* if

$$H(P, f_t) \cap H(Q, f_t) \neq \emptyset.$$

Thus, since P and Q have different indices, f_t is non-hyperbolic (unstable). We consider the sets

- $B(s) = \{t \in (0, s) : \; H(P, f_t) \cap H(Q, f_t) \neq \emptyset\}$ and
- $H(s) = \{t \in (0, s) : \; f_t \text{ is } \Omega\text{-stable } \}$.

In [DR92], [D95], and [D95b] are obtained open sets of arcs $(f_t)_{t\in[-1,1]}$ such that $H(P, f_t) = H(Q, f_t)$ for all small $t > 0$. Thus, in this case, there is t_0 so that $(0, t_0] = B(t_0)$ and we say that the cycle is *robustly non-hyperbolic* (after the bifurcation).

However, [DR97] gives open sets of arcs arcs with density of hyperbolicity close to 1. Given any $\epsilon > 0$ there is an arc $(f^{\epsilon}_t)_{t\in[-1,1]}$ with *density of hyperbolicity* larger than $1 - \epsilon$ at the bifurcation, that is,

$$\liminf_{s\to 0^+} \frac{|H(s)|}{s} > 1 - \epsilon,$$

where $|A|$ denotes the Lebesgue measure of A. On the other hand it was shown that

$$\liminf_{s\to 0^+} \frac{|B(s)|}{s} > 0,$$

thus the bifurcation value is not a point of full density of hyperbolic dynamics. In fact, in [DR01] it is proved that the occurrence of non-hyperbolicity has persistent character in the unfolding of heterodimensional cycles, i.e., the set $B(s)$ has always positive relative density at $t = 0$ for all $s > 0$.

Typically $H(s)$ is given by a collection of disjoint intervals. For all $t \in H(s)$ the homoclinic classes of P and Q are hyperbolic and disjoint. Understand the boundary of $H(s)$, that is, the transition from hyperbolic to non-hyperbolic dynamics, is one of the goals of this work.

The heterodimensional cycles were first considered by Newhouse and Palis in [NP73] and were studied systematically in the series of papers [D95, D95b, DR97, DR01, DR02, DS04, DR07]. Results show a wide variety dynamic behaviors associated with the unfolding of the cycle, depending in particular on the central eigenvalues and on the geometry of the intersection of $W^s(P) \cap W^u(Q)$. In this work we consider the simplest case, in particular we assume that the central eigenvalues are real and the cycle is *connected*: the intersection $W^s(P,f) \cap W^u(Q,f)$ has a connected component γ that is f-invariant. Moreover, in our case, γ is a curve tangent to the central direction E^c and whose extremes are the saddles P and Q.

The heuristic principle in [D95, D95b, DR02] (see also [BDV05, Chapter 6]) is that the dynamics after of the unfolding of a heterodimensional cycle is mainly determined by the action of f in γ, called *central dynamics*. In this thesis, in very rough terms, we assume that the restriction of $f = f_0$ to γ is a concave function.

To be more precise we consider a one-parameter family $(f_t)_{t\in[-\epsilon,\epsilon]}$ of diffeomorphisms unfolding a heterodimensional cycle at $t = 0$ associated to hyperbolic fixed saddles $Q = (0,0,0)$ and $P = (0,1/2,0)$ of indices 2 and 1, respectively, and we assume that the semi-local dynamics satisfies the following properties.

(P1) In the cube $\mathcal{C} = [-2,2] \times [-1,1] \times [-2,2]$, the diffeomorphism $f_0 = f$ has the form

$$f(x,y,z) = (\lambda_s x, F(y), \lambda_u z),$$

where the map F is strictly increasing with F' strictly decreasing and has two fixed points in $[0,1/2]$, 0 and 1/2. We assume that

$$\lambda_s < F'(y) < \lambda_u, \text{ for all } y \in [-1,1].$$

(P2) There are $k_0 \in \mathbb{N}$ and a small neighborhood $\mathcal{U}$ of $(0,1/2,-1) \in W^u(P,f)$ such that for small $\epsilon > 0$, the arc $(f_t)_t$ satisfies $f_t = f$ in the cube $\mathcal{C}$, for all $t \in [-\epsilon,\epsilon]$, and the restriction of $f_t^{k_0}$ to $\mathcal{U}$ is of the form

$$f_t^{k_0}(x,y,z) = \left(x-1, y-\frac{1}{2}+\alpha t, z-1\right) = f^{k_0}(x,y,z) + (0,\alpha t,0),$$

with $\alpha \in \mathbb{R}\setminus\{0\}$.

Motivated by the construction in [DHRS09], for each $a>0$, we exhibit a two-parameters model family $(f_{a,t})_{t\in[-\epsilon,\epsilon]}$, replacing the map F (defining the central dynamics) in the one-parameter family defined above, by the map

$$g_a(y) = \frac{e^a y}{2e^a y + (1-2y)}.$$

Observe that $g_a(0)=0$ and $g_a(1/2)=1/2$. So each value of a determines such a family as in **(P1)** and **(P2)**. This construction allows us to give a rather transparent explanation of the dynamics in the unfolding of the cycle and to describe secondary bifurcations generated by the unfolding of the cycle. Moreover, these families have a variety of behaviors that reproduces the ones obtained in several papers (see Figure 2).

Theorem 1. *The dynamics of $f_{a,t}$ in the neighborhood of the cycle $\mathcal{W}_a$ satisfies the following properties:*

(A) Robustly non-hyperbolic dynamics: *for $a\in(0,\log 2)$, there is $t_0(a)>0$ small, such that the homoclinic class $H(P,f_{a,t})$ and $H(Q,f_{a,t})$ coincide, for all $t\in(0,t_0(a)]$, and so they are not hyperbolic. Moreover,*

$$\Lambda_{a,t} = \bigcap_{n\in\mathbb{Z}} f_{a,t}^n(\mathcal{W}_a) = H(Q,f_{a,t}).$$

(B) Persistence of non-hyperbolicity: *for $a\in(\log 2,\log 4)$ there are $t_0(a)>0$, a sequence $t_n=t_n(a)\in(0,t_0(a)]$ converging to zero as $n\to+\infty$ verifying $\lim_{n\to+\infty} t_{n+1}/t_n = e^{-a/2}$ and a sequence of intervals*

$$J(a,t_n) = [t_n(a)-\alpha_{a,t_n}, t_n(a)+\alpha_{a,t_n}]$$

such that $\Lambda_{a,t}=H(P,f_{a,t})=H(Q,f_{a,t})$ for $t\in J(a,t_n)$.

(C) *for* $a > \log 4$*, there are* $t_0(a)$ *and* $\mu^\star_{a,t_n} \in (t_{n+1}, t_n) \cap (0, t_0(a)]$ *such that*

- **(C1) Hyperbolic dynamics:** *for every parameter* $t \in \left(\mu^\star_{a,t_n}, t_n\right)$*, the resulting non-wandering set of* $f_{a,t}$ *is hyperbolic and equal disjoint union of the (non-trivial) homoclinic classes of* P *and* Q*.*
- **(C2) Collisions of homoclinic classes via saddle-node bifurcations:** *the diffeomorphism* $f_{a,\mu^\star_{a,t_n}}$ *has a saddle-node* S_{a,t_n} *such that the intersection of the homoclinic classes of* P *and* Q *is exactly the orbit of* S_{a,t_n}*.*
- **(C3)** *Moreover* $\lim_{a\to+\infty}\left(\lim_{n\to\infty}\dfrac{t_n-\mu^\star_{a,t_n}}{t_n-t_{n+1}}\right)=1.$

The next figure summarizes these results.

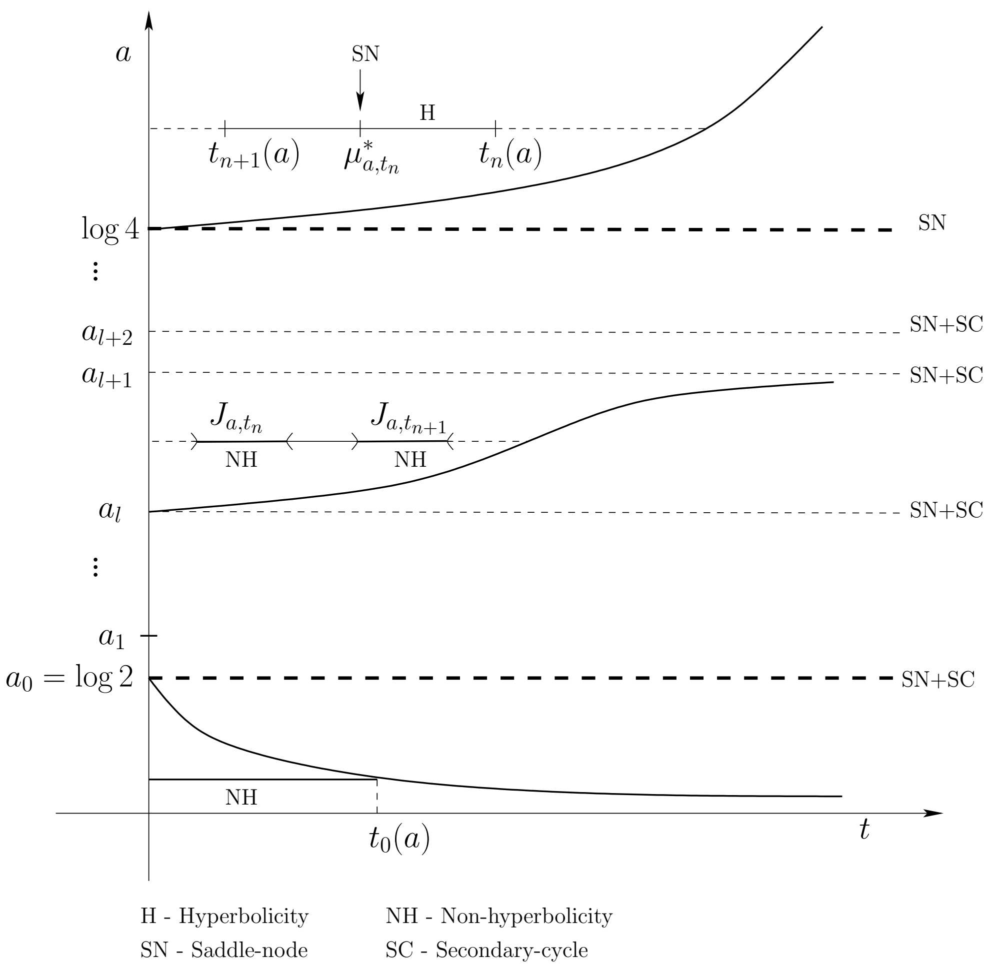

Figure 2: The dynamics of $f_{a,t}$

A key ingredient in the proof of the theorem is to reduce the dynamics in the neighborhood $\mathcal{W}_a$ of the cycle to one-dimensional dynamics. We observe that if $X \in \mathcal{C}$ then $Df_{a,t}(X)$ uniformly contracts in the $\mathbb{X}$-direction and uniformly expands in the $\mathbb{Z}$-direction, and if $X \in \mathcal{U}$ (recall that $\mathcal{U}$ is a small neighborhood of $(0, 1/2, -1)$) then $Df_{a,t}^{k_0}(X)$ is the identity. These remarks and the partial hyperbolicity of $f_{a,t}$ in $\mathcal{C}$ imply that $\Lambda_{a,t} = \bigcap_{i \in \mathbb{Z}} f_{a,t}^i(\mathcal{W}_a)$ is partially hyperbolic: the $\mathbb{X}$ and $\mathbb{Z}$- direction are hyperbolic and dominate the (central) $\mathbb{Y}$- direction.

Thus, for each $a > 0$, the dynamics of $f_{a,t}$ in $\Lambda_{a,t}$ is mainly determined by the study of one-dimensional one-parameter families of *systems of iterated functions*, $\mathfrak{F}_{a,t}$, which describe the central dynamics. These one-dimensional maps are obtained considering suitable compositions of g_a, and the quotient

$$F_{1,t} : [1/2 - \delta, 1/2 + \delta] \to \mathbb{R}, \qquad \text{where } F_{1,t}(x) = x - \frac{1}{2} + t,$$

of the restriction of $f_{a,t}^{k_0}$ to $\mathcal{U}_a$ (see property **(P2)**), defined for small $t > 0$. For example, for each $a \in (0, \log 4)$, to prove the inclusion $H(P, f_{a,t}) \subseteq H(Q, f_{a,t})$ it is enough to show that the system $\mathfrak{F}_{a,t}$ satisfies a certain expanding property. The inclusion $H(Q, f_{a,t}) \subseteq H(P, f_{a,t})$ then follows using symmetric properties of g_a.

Motivated by the fact that, roughly speaking, the construction of $\mathfrak{F}_{a,t}$ is a skew-product over a shift with two symbols and that many sequences of 0's and 1's forbidden, we introduce the *skew-products*. More precisely, we consider a family of a special sort of partially hyperbolic systems called step skew-product maps, $(G_t)_{t \in [-1,1]}$, associated to the shift σ of n-symbols and diffeomorphisms $g_{0,t}, \dots, g_{n-1,t}$, $g_{i,t} : \mathbb{K} \to \mathbb{K}$, for each $i = 0, \dots, n-1$, with $\mathbb{K} = [-1, 1]$ or $\mathbb{K} = \mathbb{S}^1$, defined by

$$G_t : \Sigma_n \times \mathbb{K} \to \Sigma_n \times \mathbb{K}, \; G_t(\xi, y) = (\sigma(\xi), g_{\xi_0,t}, y),$$

where $\xi = (\xi_i)_{i \in \mathbb{Z}}$. The space Σ_n is the base of this product, while the set $\mathbb{K}$ is the fiber.

Assume that for $t = 0$, the map g_0 is a concave function on $[-\iota, 1/2 + \iota]$, for some $\iota > 0$, with two fixed points, a repeller 0 and an attractor 1/2. Thus, the map $G_0 = G$ has two hyperbolic fixed points $Q = (0^{\mathbb{Z}}, 0)$ expanding and $P = (0^{\mathbb{Z}}, 1/2)$ contracting, and assume that there is $X \in \Sigma_n \times \mathbb{K}$ such that $X \in W^u(P, G) \cap W^s(Q, F)$. Therefore, G_0 has a heterodimensional cycle associated to the fixed points $P = (0^{\mathbb{Z}}, 1/2)$ and

$Q = (0^{\mathbb{Z}}, 0)$ of different indices. The set

$$\{0^{\mathbb{Z}}\} \times [0, 1/2] \subset W^s(P, G_0) \cap W^u(Q, G_0)$$

is called the *connection of the cycle*. We say that the orbits of X and $\{0^{\mathbb{Z}}\} \times [0, 1/2]$ are *heteroclinic intersections of the cycle*. The precise notions are given in Section 1.3.

A *neighborhood of the cycle* associated to the heteroclinic intersections (of the cycle) is a open set $\mathcal{V} = \mathcal{V}(X, \{0^{\mathbb{Z}}\} \times [0, 1/2])$ containing the orbits of the closure of $\{0^{\mathbb{Z}}\} \times [0, 1/2]$ and of X.

Although the study of the skew-product maps appear as a modulation of heterodimensional cycles, their study is important by itself. The role of the skew-products is similar to the one of the shift for the study of the horseshoe. As in heterodimensional cycles, the fiber dynamics are given by a system $\mathfrak{G}_t$ of iterated functions. In fact, this is an important tool of this work.

Inspired in the central dynamics of the family of diffeomorphisms $f_{a,t}$, we construct, for each $a > 0$, the one-parameter family of skew-product maps as

$$G_{a,t} : \Sigma_2 \times \left(-\frac{1}{2(e^a - 1)}, 1 \right] \to \Sigma_2 \times \mathbb{R}, \quad \text{with } t \in (0, \epsilon) \text{ and } a > 0,$$

such that $g_{0,t} = g_a$ and $g_{1,t}(x) = b(x - 1/2) + ct$, for all $x \in [1/2 - \zeta, 1/2 + \zeta]$. Note that, for each $a > 0$, $G_{a,0}$ has a heterodimensional cycle associated to the fixed points $P = (0^{\mathbb{Z}}, 1/2)$ and $Q = (0^{\mathbb{Z}}, 0)$.

Set $T := \{t > 0 : W^u(P, G_{a,t}) \cap W^s(Q, G_{a,t}) \neq \emptyset\}$. We observe that the parameters $t \in T$ corresponds to *secondary cycles*, that is, parameters $t > 0$ such that $G_{a,t}$ has a heterodimensional cycle associated to P and Q.

Theorem 2. *Consider the arc of skew-product maps* $(G_{a,t})_{t \in [-1,1]}$ *above. The dynamics of* $G_{a,t}$ *satisfies the following properties:*

(A) *For each* $0 < a < \log 2$ *there is* $t_0 = t_0(a) > 0$ *such that*

$$H_{\mathcal{V}}(P, G_{a,t}) \subseteq H_{\mathcal{V}}(Q, G_{a,t}), \ \forall t \in (0, t_0].$$

Moreover, if $t \notin T$ *then* $\Lambda_{a,t} = \bigcap_{n\in\mathbb{Z}} G_{a,t}(\mathcal{V}) = H_{\mathcal{V}}(Q, G_{a,t})$*, where* $\mathcal{V}$ *is a neighborhood of the cycle.*

(B) *For* $a \in (\log 2, \log 4)$ *there are* $t_0(a) > 0$ *and a sequence* $t_n(a) \in (0, t_0(a)]$ *converging to zero as* $n \to +\infty$ *and a sequence of intervals*

$$J(a, t_n) = [t_n(a) - \alpha_{a,t_n}, t_n(a) + \alpha_{a,t_n}]$$

such that $H_{\mathcal{V}}(P, G_{a,t}) = H_{\mathcal{V}}(Q, G_{a,t})$ *for* $t \in J(a, t_n)$*, and* $\Lambda_{a,t_n} = H_{\mathcal{V}}(Q, G_{a,t})$ *for* $t \in J(a, t_n) \setminus T$*.*

Note that the sequence $(t_n)_n$ remains fixed throughout the work. The reason why, for $G_{a,t}$, we need to consider $t \notin T$ to conclude that $\Lambda_{a,t} = H(Q, G_{a,t})$ is the following: for the family of skew-product maps $(G_t)_{t\in[-1,1]}$, we have

$$\big(W^u(P, G_t) \cap W^s(Q, G_t)\big) \cap H_{\mathcal{V}}(Q, G_t) = \emptyset$$

and $W^u(P, G_t) \cap W^s(Q, G_t) \subset \Lambda_{a,t}$. This is not true for heterodimensional cycles, as we can observe in Theorem 1.

After a first chapter, where we introduce some notations and definitions on heterodimensional cycles and skew-product maps, in Chapter 2 we construct, for each $a > 0$, the arcs of skew-product maps $(G_{a,t})_{t\in[-\epsilon,\epsilon]}$ and of diffeomorphisms $(f_{a,t})_{t\in[-\epsilon,\epsilon]}$. First, we analyze the system of iterated function $\mathfrak{G}_{a,t}$ for $t > 0$ and $a \in (0, \log 2)$, then we prove **(A)** of Theorem 2 and we study the system $\mathfrak{G}_{a,t}$ for $a \in (\log 2, \log 4)$, obtaining the non-hyperbolicity for $t \in J(a, t_n)$. Moreover, under an expanding condition **(EC)**, one also gets $H(Q, G_t) \subset H(P, G_t)$. Finally, using similar arguments, we conclude **(A)** and **(B)** of Theorem 1.

In Chapter 3, we consider the one-parameter family of skew-product maps $(G_t)_{t>0}$ and the arc of diffeomorphisms $(f_t)_{t>0}$ (see **(P1)** and **(P2)** above). The same results are obtained for these two families. First we prove that the periodic points of G_t in Λ_t are contained in $H_{\mathcal{V}}(P_t, G_t) \cup H_{\mathcal{V}}(Q_t, G_t)$, which implies that $G_t|_{\Lambda_t}$ has at most two homoclinic classes. Then, we state a sufficient condition to prove the existence of dense orbits for the system $\mathfrak{G}_t$.

An open subset K of $\mathrm{Diff}^1(M)$ has a *super-exponential growth for the number of periodic points* if for every arbitrary sequence of positive integers $a = (a_n)_{n=1}^{\infty}$, there is a residual subset $\mathcal{R}(a)$ of K such that

$$\limsup_{n\to\infty} \# \frac{\mathrm{Per}_n(h)}{a_n} = \infty, \quad \text{for every diffeomorphism } h \in \mathcal{R}(a),$$

where $\mathrm{Per}_n(h)$ denotes the number of isolated periodic points of period n of h. In [BDF08] it is proved that there is a residual subset $S(M)$ of $Diff^1(M)$ of diffeomorphisms h such that, for every $h \in S(M)$, any homoclinic class of h containing hyperbolic saddles of different indices has super-exponential growth of the number of periodic points. In the opposite direction, we prove the next result.

Proposition 1. *For all $t > 0$ small enough, there is $m_0 = m_0(t)$ such that*

$$Per_m(f_t|_{\Lambda_t}) \leq 2^m, \quad \text{for all } m \geq m_0.$$

In particular, for each $a > 0$ and $t > 0$ small enough, the number of periodic points of period n of $f_{a,t}$ in $\Lambda_{a,t}$ grows at most exponentially fast. Recall that, for $(a,t) \in (0, \log 2) \times (0, t_0(a))$ or $(a,t) \in (\log 2, \log 4) \times J(a, t_n)$, both homoclinic classes $H(P, f_t)$ and $H(Q, f_t)$ contains points of indices 1 and 2, being non-hyperbolic.

In [DG09] it is proved that there is a residual subset $S \in Diff^1(M)$ such that, for every $h \in S$, any homoclinic class of h containing saddles of different indices also contains an uncountable support of an invariant ergodic non-hyperbolic (one of the associated Lyapunov exponents is equal to zero) measure of h. We see that the skew-product maps $G_{a,t}$ (and the diffeomorphisms $f_{a,t}$) have a non-hyperbolic invariant ergodic measure with an uncountable support, for $(a,t) \in (0, \log 2) \times (0, t_0(a))$ or $(a,t) \in (\log 2, \log 4) \times J(a, t_n)$. Moreover, under *Non-hyperbolicity hypothesis*, one also gets the same conclusion. In fact, the maps considered above satisfy these conditions. This is de main result of Chapter 4.

Theorem 3. *For every $t > 0$ small enough, if G_t satisfies the Non-hyperbolicity hypothesis, then the map G_t has a non-hyperbolic invariant ergodic measure with an uncountable support.*

It is important to refer that, to prove the hyperbolicity for the arc $(f_{a,t})$ before the unfolding of the saddle-node, we use the existence of a filtration (see Definition 1.3) that allows us to control the dynamics of $f_{a,t}$ after the first bifurcation at $t = 0$. In fact, to get the hyperbolicity of the resulting non-wandering set it is not enough to see that the homoclinic classes of P and Q are disjoint sets. Thus, under some conditions on the global dynamics of the family $G_{a,t}$ and putting the fiber $\mathbb{S}^1$ instead of $[-1,1]$, in the last chapter we construct a new family of skew-product maps $\left(\tilde{G}_{a,t}\right)_{t\in[-1,1]}$ unfolding a heterodimensional cycle at $t=0$ and we prove the hyperbolicity of the non-wandering set for $t \in (\mu^{\star}_{a,t_n}, t_n)$, with $\mu^{\star}_{a,t_n}$ as in Theorem 1.

Chapter 1

Preliminary definitions and results

In this chapter we give relevant background definitions and concepts concerning basic notions in dynamical systems that play an important role in our study of heterodimensional cycles.

First, we review some properties about spectral decomposition, hyperbolicity and structural stability. Then, we describe a special family of diffeomorphisms unfolding a heterodimensional cycle and present the one-dimensional map F giving the central dynamics. Afterwards, we construct a one-parameter system of iterated functions describing the central dynamics after the bifurcation. Finally, we introduce a special sort of partially hyperbolic systems, called *skew-product maps*, exhibiting heterodimensional cycles.

1.1 Hyperbolicity and stability

In this section, we present fundamental concepts of the dynamical systems theory such as spectral decomposition, hyperbolicity, topological conjugacy, structural stability, and filtration.

Throughout this work, let M denote a compact, connected, and boundaryless n-dimensional Riemannian manifold, and $f : M \to M$ a diffeomorphism.

Definition 1.1. *A point $P \in M$ is* periodic *under f if there exists some $m \in \mathbb{N}$, called the* period *of P, such that $f^m(P) = P$ and $f^j(P) \neq P$ for $0 < j < m$. If P has period one, then it is called a* fixed point.

We say that a periodic point P of f is a *hyperbolic fixed point* of f if $f(P) = P$ and if $Df(P)$ has no eigenvalue of norm 1. If P is a hyperbolic fixed point and there are eigenvalues λ, β of $Df(p)$ satisfying $|\beta| > 1 > |\lambda|$, then P is called a *saddle.*

For a hyperbolic fixed point P and a neighborhood U of it, the *local stable* and the *local unstable manifolds* are defined respectively as:

$$W^s_{loc}(P, f) = \{x \in U : \ f^j(x) \in U \text{ for all } j \in \mathbb{N}\} \text{ and}$$
$$W^u_{loc}(P, f) = \{x \in U : \ f^{-j}(x) \in U \text{ for all } j \in \mathbb{N}\}.$$

The *stable and unstable manifolds* are respectively:

$$W^s(P, f) = \bigcup_{j\geq 0} f^{-j}\left(W^s_{loc}(P, f)\right) \text{ and}$$
$$W^u(P, f) = W^s(p, f^{-1}) = \bigcup_{j\geq 0} f^{j}\left(W^u_{loc}(P, f)\right).$$

According to the Invariant Manifold Theorem (see [HPS77]) the sets $W^s(P, f)$ and $W^u(P, f)$ are injectively immersed submanifolds of M, with the same differentiability of f, and with dimensions equal, respectively, to the number of eigenvalues of $Df(P)$ with norm smaller, bigger, than one.

A point $X \neq P$ is called *homoclinic* to P if $X \in W^s(P, f) \cap W^u(P, f)$, that is, if $X \neq P$ and $\lim_{i\to\pm\infty} f^i(X) = P$. If $W^s(P, f)$ and $W^u(P, f)$ intersect transversally at X, i.e. if

$$T_X M = T_X W^s(P, f) \oplus T_X W^u(P, f),$$

then X is called a *transverse homoclinic point.* One can give the corresponding definitions for periodic points of f, because they are fixed points of f^k.

The *homoclinic class* of a hyperbolic saddle P, denoted by $H(P, f)$, is the closure of the transversal homoclinic points of f associated to P. A homoclinic class is *trivial* if it consists of a single orbit, and two hyperbolic periodic orbits are *homoclinically related* if the stable manifold of each point intersects transversally the unstable manifold of the other. Using this homoclinic relation introduced by Smale, we can also define the homoclinic class of a periodic saddle P as the closure of the set of saddles which are homoclinically related to P. We observe that two saddles which are homoclinically related have the same Morse index, that is, the same dimension of the unstable manifold, denoted by

$$\text{index}(P) := \dim W^u(P, f).$$

Thus, the set of saddles of the same index as P is dense in the whole homoclinic class of P.

We observe that, for a hyperbolic fixed point P, there is a Df- invariant splitting $T_P M = E^s \oplus E^u$, where E^s and E^u denotes the eigenspaces of $Df(P)$, corresponding to the eigenvalues whose norm is less than 1 and greater that 1, respectively. Thus $Df|_{E^s}$ is contracting and $Df|_{E^u}$ is expanding. Next, we extend this notion to an f-invariant compact set $\Lambda \subset M$.

An f-invariant compact set Λ is called *hyperbolic* if the tangent bundle over Λ admits a continuous splitting $T_\Lambda M = E^s \oplus E^u$, preserved by Df, and there exists numbers $C > 0$ and $\lambda < 1$ such that for all $n \in \mathbb{N}$

$$\max\left\{\| \left(Df|_{E^s}\right)^n \|, \| \left(Df|_{E^u}\right)^{-n} \|\right\} \leq C\lambda^n.$$

Note that we can choose a Riemannian metric on the manifold M so that $C = 1$.

A *dominated splitting* on Λ is a continuous $Df-$ invariant splitting $T_\Lambda M = E \oplus F$ such that there are $m \in \mathbb{N}$ and $0 < \lambda < 1$ satisfying

$$\|(Df^m)|_{E(X)}\|\|(Df^{-m}|_{F(f^m(X))})\| < \lambda$$

for all $X \in \Lambda$. We say that $T_\Lambda M = F_1 \oplus F_2 \oplus F_3$ is a *double dominated splitting* if both $F_1 \oplus (F_2 \oplus F_3)$ and $(F_1 \oplus F_2) \oplus F_3$ are dominated splittings.

We say that Λ is a *partially hyperbolic set* with one-dimensional center manifold of f if there exists a continuous $Df-$ invariant splitting

$$T_\Lambda M = E^s \oplus E^c \oplus E^u$$

with $\dim E^c(x) = 1$ $(x \in \Lambda)$, satisfying the following properties:

- the splitting is double dominated,
- both subbundles E^s and E^u are not zero and
- $Df|_{E^s}$ is uniformly contracting and $Df|_{E^u}$ is uniformly expanding.

One of the goals of the dynamical systems theory is to understand the behavior of the sequences $\{f^i(X)\}_{i\in\mathbb{Z}}$, for every $X \in M$. Thus, for $X \in M$, we define the *α-limit and ω-limit sets* as

$$\alpha(x) := \{Y \in M | \exists n_i \to -\infty \text{ such that } f^{n_i}(X) \to Y\} \text{ and}$$
$$\omega(x) := \{Y \in M | \exists n_i \to +\infty \text{ such that } f^{n_i}(X) \to Y\},$$

respectively, and the *positive and negative limit sets* as

$$L^+(f) := \overline{\bigcup_{X\in M} \omega(X)} \text{ and } L^-(f) := \overline{\bigcup_{X\in M} \alpha(X)},$$

respectively, denoting the limit set, i.e., the union of $L^+(f)$ and $L^-(f)$, by $L(f)$.

Another important notion is the following. A point X is called *non-wandering* for f if for every neighborhood U of X there is $n > 0$ such that $f^n(U)$ intersects U. Naturally, $f^{-n}(U)$ also intersects U, which means that there is $Y \in U$ such that $f^{-n}(Y) \in U$. Clearly, all α, ω- limits points are non-wandering points, as well as the homoclinic points. The set of all non-wandering points for f is called the *non-wandering set* and is denoted by $\Omega(f)$.

Definition 1.2. *The diffeomorphism $f : M \to M$ is called* uniformly hyperbolic*, or* Axiom A*, if $\Omega(f)$ is a hyperbolic set and $Per(f)$ is dense in $\Omega(f)$, where $Per(f)$ indicates the set of periodic points of f.*

In [S70], Smale proved that an Axiom A diffeomorphism f decomposes the non-wandering set $\Omega(f)$ as a finite pairwise disjoint union, i.e.,

$$\Omega(f) = \bigcup_{i=1}^{m} \Lambda_i,$$

called the *spectral decomposition of* $\Omega(f)$, where each set Λ_i is $f-$ invariant, transitive (i.e. it has a dense orbit), local maximal (i.e. there exists a neighborhood $\mathcal{U}_i$ of Λ_i such that $\Lambda_i = \bigcap_{n\in\mathbb{Z}} f^n(\mathcal{U}_i))$, and compact. Moreover, $\Lambda_i = H(P, f)$, where P is any point of $\Lambda_i \cap Per(f)$ and we have index(P_1) = index(P_2), for every $P_1, P_2 \in \Lambda_i \cap Per(f)$. The sets Λ_i and $\mathcal{U}_i$ in the spectral decomposition are called, respectively, *basic sets* and *resulting neighborhood* of Λ_i. Note that, a priori, a homoclinic class may contain periodic points having different indices, once being non-hyperbolic.

The basic sets are persistent under C^k- small perturbations, i.e., given a basic set Λ and an isolating neighborhood U of Λ, for any g C^k- close to f, $k \geq 1$, the set

$$\Lambda(g) = \bigcap_{n\in\mathbb{Z}} g^n(U)$$

is hyperbolic and there is a homeomorphism $h : \Lambda \to \Lambda(g)$ such that $g \circ h(x) = h \circ f(x)$ for every $x \in \Lambda$ (see [HP70]). The map h is called *conjugation* and the set $\Lambda(g)$ is the *"smooth" continuation of* Λ. Moreover, we say that a diffeomorphism f is $C^k - \Omega$-*stable* if for any g C^k- close to f, $f|_{\Omega(f)}$ is conjugate to $g|_{\Omega(g)}$.

Now let f be an Axiom A diffeomorphism and let $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_k$ be its spectral decomposition. A collection of basic sets $\Lambda_{i_1}, \ldots, \Lambda_{i_k}$ is called a *cycle* if there exist points $x_1, y_1 \in \Lambda_{i_1}, \ldots, x_k, y_k \in \Lambda_{i_k}$, with not all $i_1, \ldots, i_k$ equal, such that

$$W^u(y_1, f) \cap W^s(x_2, f) \neq \emptyset, \ \ldots, \ W^u(y_k, f) \cap W^s(x_1, f) \neq \emptyset.$$

The Ω-stability conjecture of Palis-Smale in [PS70] states that a diffeomorphism f is C^k stable if and only if it is an Axiom A and it satisfies the no cycles property. Smale's proved that these properties are sufficient (see Smale's Stability Theorem [S70]) and Palis proved that this conditions are necessary in the C^1 topology (see [P88]). The proof involves the following notion:

Definition 1.3. *A* filtration *for a diffeomorphism $f : M \to M$ is a finite family $M_1, M_2, \ldots, M_k$ of submanifolds with boundary and with the same dimension as M, such that $M_1 = M$ and M_{i+1} is contained in the interior of M_i, for every $1 \leq i < k$, and $f(M_i)$ is contained in the interior of M_i for all $1 \leq i \leq k$. The open sets $L_i = \operatorname{int}(M_i \setminus M_{i+1})$ are the* levels of the filtration *(set $M_{k+1} = \emptyset$).*

1.2 Heterodimensional Cycles

In this section, we construct a one-parameter family $(f_t)_{t\in[-1,1]}$ of diffeomorphisms unfolding at $t = 0$ a heterodimensional cycle. The study of the semi-local dynamics of f_t will be reduced to the analysis of a system $\mathfrak{F}_t$ of iterated functions that describe the dynamics of f_t in the central direction.

1.2.1 The model one-parameter family

Here, we describe a model arc unfolding a heterodimensional cycle.

Consider a diffeomorphism $f : M \to M$ having two hyperbolic periodic points Q and P. We say that there is a *cycle related to* P *and* Q if $W^u(Q,f) \pitchfork W^s(P,f) \neq \emptyset$ and $W^u(P,f) \cap W^s(Q,f) \neq \emptyset$. If index($P$)=index($Q$) the cycle is *equidimensional* and *heterodimensional* otherwise, i.e. if the periodic saddles P and Q have different dimensions of their unstable subspaces. Note that, for having a heterodimensional cycle, the dimension of M must be at least equal to three. Moreover, for heterodimensional cycles, in the general case $W^u(P,f)$ and $W^s(Q,f)$ have a quasi-transverse intersection:

$$T_X W^s(Q,f) \cap T_X W^u(P,f) = \{0\}$$

for every intersection point $X \in W^s(Q,f) \cap W^u(P,f)$.

Let $(f_t)_{t\in[-1,1]}$ be a parameterized family of diffeomorphisms and denote by P_t and Q_t the continuations for f_t of the hyperbolic periodic points P and Q. We say that the arc $(f_t)_{t\in[-1,1]}$ *unfolds generically* a heterodimensional cycle if there is a heterodimensional cycle associated to P and Q, for $t = 0$, and there are open disks $K_t^u(P) \subset W^u(P_t, f_t)$ and $K_t^s(Q) \subset W^s(Q_t, f_t)$, depending continuously on t, such that

$$K_0^u(P) \cap K_0^s(Q) = \{X_0\}$$

where X_0 is a point of quasi-transverse intersection, and the distance between $K_t^u(P)$ and $K_t^s(Q)$ increases with positive speed when t increases.

For notational simplicity, we will consider a heterodimensional cycle in $\mathbb{R}^3$, the extension to higher dimensions is straightforward. We also add the assumption that the periodic points P and Q in the cycle are fixed points.

Consider a diffeomorphism f with a heterodimensional cycle associated to the saddle fixed points $Q = (0,0,0)$ and $P = (0,1/2,0)$ of indices 2 and 1, respectively. We assume that the cycle verify the conditions (a), (b), (c) and (d) below.

(a) Partial hyperbolic (semi-local) dynamics of the cycle

In the cube $\mathcal{C} = [-2,2] \times [-1,1] \times [-2,2]$, the diffeomorphism f has the form

$$f(x,y,z) = (\lambda_s x, F(y), \lambda_u z), \tag{1.2.1}$$

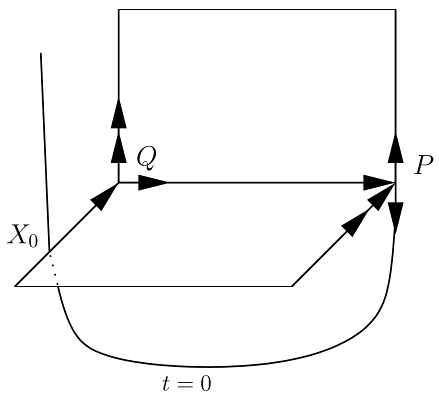

Figure 1.1: The heterodimensional cycle

where the map F is strictly increasing with F' strictly decreasing and has two fixed points in $[0, 1/2]$, 0 and 1/2.

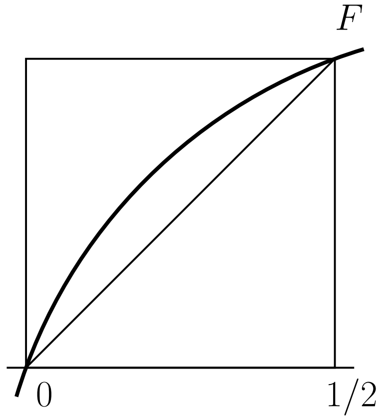

Figure 1.2: The central map F

We assume that $\lambda_s < F'(y) < \lambda_u$, for all $y \in [-1, 1]$. This assumption allows us to reduce the study of the dynamics in a neighborhood of the cycle to one-dimensional dynamics. Denote by $\beta > 1$ and $0 < \lambda < 1$ the derivative of F at 0 and 1/2, respectively.

Observe also that, by definition of the diffeomorphism f,

$$[-2, 2] \times \{(0, 0)\} \subset W^s(Q, f) \text{ and } \{0\} \times \left[0, \tfrac{1}{2}\right) \times [-2, 2] \subset W^u(Q, f) \text{ and}$$
$$\{(0, 0)\} \times [-2, 2] \subset W^u(P, f) \text{ and } [-2, 2] \times \left(0, \tfrac{1}{2}\right] \times \{0\} \subset W^s(P, f).$$

Consequently $W^s(P, f)$ and $W^u(Q, f)$ meet transversely along the *connection curve* $\gamma = \{0\} \times (0, 1/2) \times \{0\}$.

(b) Existence and unfolding of the cycle

The cycle: There are $k_0 \in \mathbb{N}$ and a small neighborhood $\mathcal{U}$ of $(0, 1/2, -1) \in W^u(P, f)$ such that the restriction of f^{k_0} to $\mathcal{U}$ is the translation

$$f^{k_0}(x, y, z) = \left(x - 1, y - \frac{1}{2}, z + 1\right). \tag{1.2.2}$$

Therefore $f^{k_0}\,(0, 1/2, -1) = (-1, 0, 0) \in W^s(Q, f)$ and, by construction, $X_0 = (-1, 0, 0)$ is a quasi-transverse heteroclinic point, that is,

$$W^u(P, f) \cap W^s(Q, f) \supseteq \bigcap_{n \in \mathbb{Z}} f^n(X_0)$$

and $T_{X_0}W^u(P, f) + T_{X_0}W^s(Q, f) = T_{X_0}W^u(P, f) \oplus T_{X_0}W^s(Q, f) = \mathbb{XZ}$.

The unfolding of the cycle: For small $\epsilon > 0$, considerer an arc $(f_t)_{t\in[-\epsilon,\epsilon]}$ of diffeomorphisms coinciding with f in the cube $\mathcal{C}$ and such that the restriction of $f_t^{k_0}$ to $\mathcal{U}$ is of the form

$$\begin{aligned} f_t^{k_0}(x, y, z) &= \left(x - 1, y - \tfrac{1}{2} + t, z - 1\right) \\ &= f^{k_0}(x, y, z) + (0, t, 0). \end{aligned}$$

So, for $t > 0$, $\{(-1, t)\} \times [-2, 2] \subset W^u(P, f_t)$ and $X_t = (-1, t, 0)$ is a transverse homoclinic point of P (for f_t). Similarly, $Y_t = (-1, 0, 0)$ is a transverse homoclinic point of Q. Thus, for $t > 0$, $H(P, f_t)$ and $H(Q, f_t)$ are both non trivial.

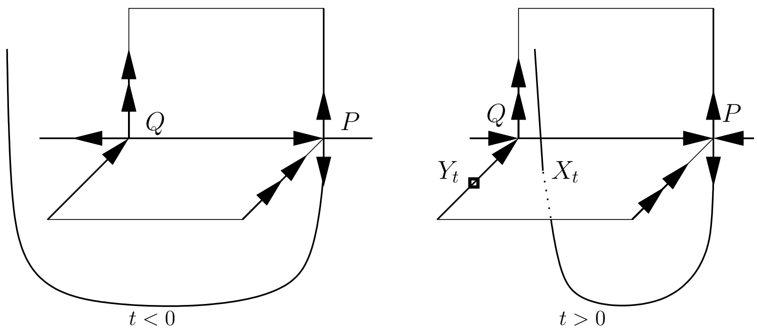

Figure 1.3: The unfolding of the cycle

(c) Existence of filtrating neighbourhood of the cycle

We say that $\mathcal{W}$ is a *neighborhood of the cycle* if $\mathcal{W}$ is a neighborhood of all elements involved in the cycle: the fixed points P and Q, the connection γ and the f-orbit of the heteroclinic point $X_0 = (-1, 0, 0)$.

Let $\mathcal{W}$ be a small neighborhood of the cycle and assume that it is a *filtrating neighborhood* of $f = f_0$: there are compact manifolds M_1 and M_2 with boundary of the same dimension as M, $M_1 \subset M_2$, such that $\mathcal{W} = M_1 \setminus int(M_2)$.

This implies that if $x \in \mathcal{W}$ and $f(x) \notin \mathcal{W}$, then $f^n(x) \notin \mathcal{W}$, for every $n \in \mathbb{N}$. Analogously, if $x \in \mathcal{W}$ and $f^{-1}(x) \notin \mathcal{W}$, then $f^{-n}(x) \notin \mathcal{W}$, for every $n \in \mathbb{N}$.

Let

$$\Lambda_t = \bigcap_{i \in \mathbb{Z}} f_t^i(\mathcal{W})$$

be the maximal f_t-invariant in $\mathcal{W}$. Using the filtration and the choice of $\mathcal{W}$ it follows that the *resulting non-wandering set*

$$\Omega(f_t)' = \Omega(f_t) \cap \mathcal{W}$$

is contained in Λ_t. In particular, $H(P, f_t) \cup H(Q, f_t) \subset \Lambda_t$.

(d) Invariant foliations Let $X = (x_0, y_0, z_0) \in \mathcal{C}$. We define by

$$\begin{aligned} F^s(X) &:= \{(x, y, z) \in \mathcal{C} : y = y_0, z = z_0\}, \\ F^u(X) &:= \{(x, y, z) \in \mathcal{C} : x = x_0, y = y_0\}, \text{ and} \\ F^c(X) &:= \{(x, y, z) \in \mathcal{C} : x = x_0, z = z_0\}, \end{aligned}$$

the strong stable, strong unstable and central leaves, respectively. This defines the strong stable, strong unstable and central foliations on $\mathcal{C}$. We extend the three foliations, via f_t, to $\mathcal{W}$. By construction, one has the f_t-invariance of the foliations in the following sense: if $X \in \mathcal{C}$ and $f_t(X) \in \mathcal{C}$, then $f_t(F^s(X)) \subset F_t^s(f_t(X))$, $f_t(F^u(X)) \supset F^u(f_t(X))$ and $f_t(F^c(X)) \cap F^c(f_t(X))$ is a neighborhood of $f_t(X)$ in $F^c(f_t(X))$; if $X \in \mathcal{C}$, $f_t(X) \notin \mathcal{C}$ and $f_t^{k_0}(X) \in \mathcal{C}$ (i.e. $X \in \mathcal{U}$), then the connected component in $\mathcal{C}$ of $f_t^{k_0}(F^\ell(X))$ that contains $f_t^{k_0}(X)$ is a subset of $F^\ell(f_t^{k_0}(X))$, $\ell = s, u, c$.

Note that if $X \in \mathcal{C}$ then $Df_t(X)$ uniformly contracts in the $\mathbb{X}$-direction and uniformly expands in the $\mathbb{Z}$-direction, and if $X \in \mathcal{U}$ (recall that $\mathcal{U}$ is a small neighborhood of $(0, 1/2, -1)$) then $Df_t^{k_0}(X)$ is the identity. These remarks and the partial hyperbolicity of f_t in $\mathcal{C}$ imply that Λ_t is partially hyperbolic: the $\mathbb{X}$ and $\mathbb{Z}$- direction are hyperbolic and dominate the (central) $\mathbb{Y}$- direction. Thus the dynamics of f_t is mainly determined by its central dynamics and the limit dynamics in the central dynamics is given by a one-parameter family of iterated function systems defined on an interval whose generators are the restriction of f to the $\mathbb{Y}$-axis, denoted by F, and the function

$$F_{1,t} : [1/2-\delta, 1/2+\delta] \to \mathbb{R}, \quad F_{1,t}(y) = y - \frac{1}{2} + t,$$

of the restriction of $f_t^{k_0}$ to the central coordinates of $\mathcal{U}$, defined for small $t > 0$.

1.2.2 Returns and central dynamics

In this subsection we describe the itineraries of points in the resulting non-wandering set and introduce a system $\mathfrak{F}_t$ of iterated functions describing the central dynamics. We also define a return map R_t in a fundamental domain of F with infinitely many discontinuities.

Given $t > 0$ small enough, denote by $D_t^P := [1/2-t, F(1/2-t)]$ the fundamental domain of F at distance t from P. Let $d_t := F^{-N_t}(1/2-t)$ be the unique backward iterate of $1/2-t$ in the interval $[t, F(t)]$ and $D_t^Q := [d_t,\ F(d_t)]$. By construction, $F^{N_t}(D_t^Q) = D_t^P$ and the map F^{N_t} is defined as the transition from 0 to 1/2 (see Figure 1.4).

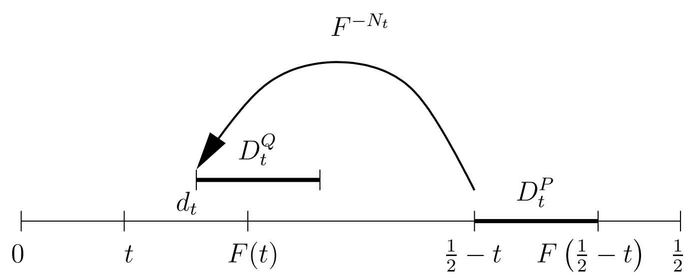

Figure 1.4: The construction of D_t^Q

Consider the increasing map h_t defined by

$$h_t : D_t^Q \to [0,t)\,, \quad h_t(y) = F_{1,t} \circ F^{N_t}(y) = F^{N_t}(y) - \frac{1}{2} + t. \tag{1.2.3}$$

Let m_0 be the first natural number which verifies $F^{m_0} \circ h_t(F(d_t)) \in D_t^Q$. Given a point $y \in (d_t,\ F(d_t)]$, let $m = m(y) \geq m_0$ be the minimum m with $F^{m(y)} \circ h_t(y) \in D_t^Q$.

This defines a return map to the fundamental domain D_t^Q:

$$R_t : (d_t, F(d_t)] \to D_t^Q, \quad R_t(y) = F^{m(y)} \circ h_t(y),$$

The map R_t has (infinitely many) discontinuities where the lateral derivatives are well defined. For each $m \geq m_0$, define $d_m^t \in D_t^Q$ by $F^m \circ h_t(d_m^t) = d_t$. By construction, the sequence $(d_m^t)_{m \geq m_0}$ corresponds to the discontinuities of R_t. Consequently $(d_m^t)_{m \geq m_0}$ is a decreasing sequence that accumulates at the point d_t. In the interior of each interval $[d_{m+1}^t, d_m^t]$, $m \geq m_0$, R_t is continuous, onto and strictly increasing map (see Figure 1.5). We continuously extend R_t to the whole interval $[d_{m+1}^t, d_m^t]$, obtaining a bi-valuated map with $R_t(d_m^t) \in \{d_t, F(d_t)\}$. More, since F' is a strictly decreasing map, we have that R_t' is also strictly decreasing.

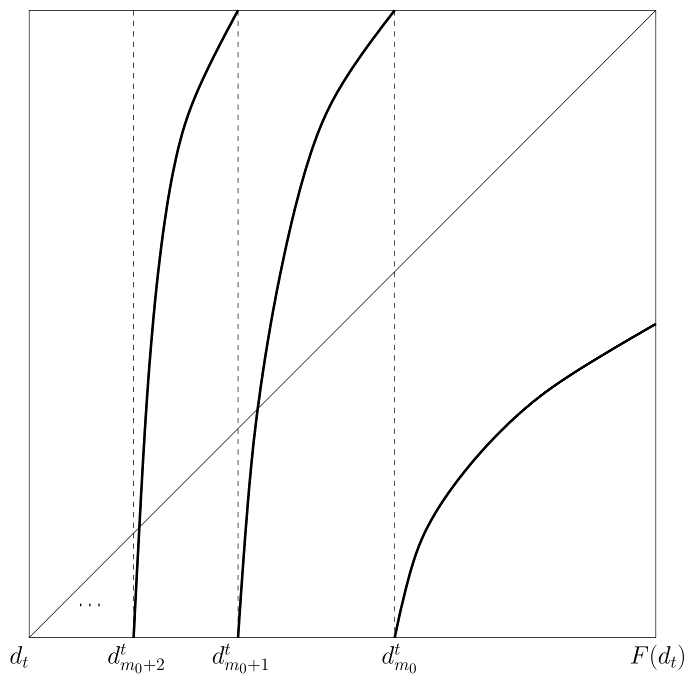

Figure 1.5: The return map R_t

Let $X \in \Lambda_t \cap \Delta_t^Q$, where $\Delta_t^Q := [-2, 2] \times D_t^Q \times [-2, 2]$. If $X \notin W^s(P) \cap W^s(Q)$, then it forward orbit returns infinitely many times to Δ_t^Q. We say that X has a return of type (q, p) if $f_t^{q+k_0+p+N_t}(X)$ is in Δ_t^Q. The map R_t defines the central y-coordinate for the points in Δ_t^Q having a return of type $(q, 0)$. Note that the point X could have different returns to Δ_t^Q.

Motivated by the last observation, we consider the parameterized family of maps

$(\Phi_t^{q,p})_{q,p\geq 0}$ defined by

$$\Phi_t^{q,p}(y) = F^q \circ F_{1,t} \circ F^{p+N_t}(y),\ q,\ p \geq 0 \tag{1.2.4}$$

defined for each pair (q,p) from $D_t^{q,p}$ to D_t^Q, where $D_t^{q,p}$ is the maximal subset of D_t^Q consisting of points y with $\Phi_t^{q,p}(y) \in D_t^Q$. Note that, from the monotonicity of F, $D_t^{q,p}$ is either empty or a closed subinterval of D_t^Q.

For a sequence $((q_k,p_k))_{k\in\mathbb{N}}$ in $\mathbb{N}_0 \times \mathbb{N}_0$ and $l \in \mathbb{N}$, let us consider the l-block $\varrho_l := [(q_1,p_1),\ldots,(q_l,p_l)]$ and the associated map

$$\Phi_t^{\varrho_l} := \Phi_t^{q_l,p_l} \circ \cdots \circ \Phi_t^{q_1,p_1} \tag{1.2.5}$$

which is defined in $D_t^{\varrho_l} = D_t^{q_1,p_1} \cap (\Phi_t^{q_1,p_1})^{-1}(D_t^{q_2,p_2}) \cap \cdots \cap \left(\Phi_t^{q_{l-1},p_{l-1}}\right)^{-1}(D_t^{q_l,p_l})$.

Let $X = (x_0,y_0,z_0) \in \Delta_t^Q$. If X has a (q,p) return, then $y_0 \in D_t^{q,p}$ and the y-coordinate of $f_t^{q+k_0+p+N_t}(X)$ is $\Phi_t^{q,p}(y_0)$, where N_t is the number of iterations from Δ_t^Q to $\Delta_t^P := [-2,2] \times D_t^P \times [-2,2]$, and k_0 is the number of iterations needed to go from $\mathcal{U}$ to $\mathcal{C}$. Conversely, given any point $y_0 \in D_t^{q,p}$, due to the partial hyperbolicity, there is at least one point of the form $X = (x,y_0,z) \in \Delta_t^Q$ with a (q,p)- return. If we consider the system of iterated functions $\mathfrak{F}_t$ defined as follows

$$\mathfrak{F}_t = \{\Phi_t^{q,p} : (q,p) \in \mathbb{N} \times \mathbb{N}\},$$

we can translate some relevant dynamical properties to similar properties of f_t. For example, to a attractive fixed point of $\Phi_t^{q,p}$ corresponds a periodic point of f_t of index one with period $q+k_0+p+N_t$, where q and p determine the central coordinate of the point, and to a repulsive fixed point of $\Phi_t^{q,p}$ corresponds a periodic point of f_t of index two with period $q+k_0+p+N_t$.

1.3 Skew-products and heterodimensional cycles

Recall that, the study of the dynamics of f_t in the neighborhood $\mathcal{W}$ of the cycle was reduced to the study of the dynamics in the central direction and the central dynamics of f_t is obtained considering suitable compositions of F and $F_{1,t}$. In rough terms, the construction of $\mathfrak{F}_t$ is a skew-product over a shift with two symbols and many sequences of 0's and 1's forbidden.

Here, we construct a family $(G_t)_{t\in[-1,1]}$ of skew-product maps defined over a full shift space with finitely many symbols such that G_0 has a heterodimensional cycle. In the setting of skew-product maps, we begin by stating some notations and definitions such as heterodimensional cycle, homoclinic class and neighborhood of the cycle. In this section, we follow the definitions, notations, and models introduced in [DR].

1.3.1 Notations and definitions

For $n \in \mathbb{N}$, Σ_n is the metric space of two sided infinite sequences over de alphabet $\{0, \cdots, n-1\}$, i.e. $\Sigma_n = \{0,\ 1,\ \ldots,\ n-1\}^{\mathbb{Z}}$, equipped with the metric

$$d_{\Sigma_n}(\xi, \xi') = 2^{-\min\{|n|:\xi_n \neq \xi'_n\}},$$

and $\sigma : \Sigma_n \to \Sigma_n$ is the Bernoulli shift.

We define a *skew-product* over the Bernoulli shift as a map

$$G : \Sigma_n \times M \to \Sigma_n \times M, \quad G(\xi, x) = (\sigma(\xi), g_{\xi_0}(x)), \tag{1.3.1}$$

where $g_0,\ \cdots g_n : M \to M$ are diffeomorphisms of a compact manifold M. We say that Σ_n is the base and the manifold M is the fiber of the product. Since the map G on the fiber depends only on the zeroth element of the sequence ξ in the base, the skew-product is called *step*. The skew-product defined by

$$\mathcal{G} : \Sigma_n \times M \to \Sigma_n \times M, \quad \mathcal{G}(\xi, x) = (\sigma(\xi), g_\xi(x)),$$

is called *soft* since the map in the fiber depends on the entire word ξ. For more details about soft skew-products, see for instance [GI00].

Let G be a step skew-product map with $M = \mathbb{K}$, where $\mathbb{K} = \mathbb{S}^1$ or $\mathbb{K} = [-1, 1]$. Next we introduce and make the natural adaptations of some definitions introduced in the previous sections.

Definition 1.4. *For $r, m \in \mathbb{N}_0$, consider numbers $\xi_{-r}, \ldots, \xi_0, \ldots, \xi_m \in \{0, \ldots, n-1\}$. The associated* cylinder map *is defined by*

$$g_{[\xi_0\cdots\xi_m]} : \mathbb{K} \to \mathbb{K}, \quad g_{[\xi_0\cdots\xi_m]}(x) = g_{\xi_m} \circ \cdots \circ g_{\xi_0}(x).$$

and the cylinder $[\xi_{-r}\cdots.\xi_0\cdots\xi_m]$ *is the subset of* Σ_2 *defined by*

$$[\xi_{-r}\cdots.\xi_0\cdots\xi_m]=\{(\eta_k)_{k\in\mathbb{Z}}:\eta_i=\xi_i,\quad \textit{for all } i\in\{-r,\dots,m\}\}.$$

If $\xi=(\xi_k)_{k\in\mathbb{Z}}\in\Sigma_n$ is a periodic sequence of period m of σ, then we write $\xi=(\xi_0\cdots\xi_{m-1})^{\mathbb{Z}}$. We denote by $\big((\delta_{-r}\cdots\delta_{-1})^{-\mathbb{N}}\eta_{-l}\cdots\eta_{-1}.\eta_0\cdots\eta_k(\alpha_1\cdots\alpha_m)^{\mathbb{N}}\big)$ the sequence $\xi=(\xi_i)_{i\in\mathbb{Z}}$ defined by

- $\xi_i=\eta_i$, for $i\in\{-l,\dots,k\}$;
- $\xi_{k+sm+i}=\alpha_i$, for every $i\in\{1,\dots,m\}$ and $s\geq 0$,
- $\xi_{-l-sr-i}=\delta_{-i}$ for every $i\in\{1,\dots,r\}$ and $s\geq 0$.

Moreover, we denote

$$g_{[\xi_{-r}\cdots\xi_{-1}.]}=\left(g_{\xi_{-1}}\circ\cdots\circ g_{\xi_{-r}}\right)^{-1}. \tag{1.3.2}$$

If $X=(\xi,x)\in\Sigma_n\times\mathbb{K}$ is a periodic point of G of period m, we have

$$\sigma^m(\xi)=\xi \text{ and } g_{[\xi_0\cdots\xi_{m-1}]}(x)=x.$$

Since the map is only differentiable in the fiber direction, we say that a periodic point $P=(\xi,p)$ of G of period m is *hyperbolic* if

$$g'_{[\xi_0\cdots\xi_{m-1}]}(p)\neq\pm1.$$

The hyperbolic periodic point P is of *contracting type* if this derivative has modulus less than one, otherwise this point is of *expanding type*. We say that two periodic points P and Q have the *same indices* if both points are of contracting type or both points are of expanding type, otherwise we say that the points have *different indices*.

Let $A=\big((\xi_0\cdots\xi_{m-1})^{\mathbb{Z}},a\big)$ be a periodic point of G of period m. We define the stable manifold by

$$W^s(A,G)=\{(\eta,x):\begin{cases}\eta=\big(\cdots.\eta_0\cdots\eta_k(\xi_0\cdots\xi_{m-1})^{\mathbb{N}}\big),\\ g_{[\eta_0\cdots\eta_k]}(x)\in W^s_{loc}(a,g_{[\xi_0\cdots\xi_{m-1}]})\end{cases} \tag{1.3.3}$$

where $W^s_{loc}(a, g_{[\xi_0\cdots\xi_{m-1}]})$ is the usual local stable manifold of a for the map $g_{[\xi_0\cdots\xi_{m-1}]}$. Analogously, we define unstable manifold by

$$W^u(A,G) = \{(\eta, x) : \begin{cases} \eta = \left((\xi_0\cdots\xi_{m-1})^{-\mathbb{N}}\eta_{-k}\cdots\eta_{-1}.\cdots\right), \\ g_{[\eta_{-k}\cdots\eta_{-1}.]}(x) \in W^u_{loc}(a, g_{[\xi_0\cdots\xi_{m-1}]}) \end{cases} \tag{1.3.4}$$

where $W^u_{loc}(a, g_{[\xi_0\cdots\xi_{m-1}]})$ is the usual local unstable manifold of a for the map $g_{[\xi_0\cdots\xi_{m-1}]}$.

Note that if A is of expanding type we have $W^s_{loc}(a, g_{[\xi_0\cdots\xi_{m-1}]}) = \{a\}$, therefore

$$g_{[\eta_0\cdots\eta_k]}(x) = a, \quad \text{for all } (\eta, x) \in W^s(A,G)$$

and, in similar way, if A is of contracting type we conclude that

$$g_{[\eta_{-k}\cdots\eta_{-1}.]}(x) = a, \quad \text{for all } (\eta, x) \in W^u(A,G).$$

We say that a pair of periodic points $P = (\alpha, p)$ and $Q = (\gamma, q)$ of the skew-product map G has a *heterodimensional cycle* if the stable manifold of P intersects the unstable manifold of Q and the unstable manifold of P intersects the stable manifold of Q.

Definition 1.5. *Let $P = (\alpha, p)$ be a periodic point of G of contracting (respectively expanding) type. A point $X = (\xi, x)$ is called a* transverse homoclinic point of P *if $X \in W^u(P,G) \cap W^s(P,G)$ and there is a open interval $I \subset \mathbb{K}$ such that $x \in I$ and $\{\xi\} \times I \subset W^s(P,G)$ (respectively $\{\xi\} \times I \subset W^u(P,G)$).*

The *homoclinic class* of a periodic point $P = (\alpha, p)$ of G, denoted by $H(P,G)$, is defined by the closure of the transverse intersections of the invariant sets $W^s(P,G)$ and $W^u(P,G)$, i.e.,

$$H(P,G) = \overline{W^s(P,G) \pitchfork W^u(P,G)}.$$

We say that two hyperbolic periodic points P and Q of G are *homoclinically related* if $W^s(P,G) \pitchfork W^u(Q,G) \neq \emptyset$ and $W^u(P,G) \pitchfork W^s(Q,G) \neq \emptyset$.

Given a neighborhood U of the orbit of a periodic point P, *the relative homoclinic class of P to U*, denoted by $H_U(P,G)$, is the subset of $H(P,G)$ of points whose orbit is contained in U.

Now, in the skew-product map G, we assume that g_0 is an orientation preserving (or increasing) diffeomorphism on $\mathbb{K}$ with two fixed points, a repeller 0 and an attractor 1/2 and that g_0' is strictly decreasing on $[-\iota, 1/2+\iota]$, for some $\iota > 0$. Then, the map G has two hyperbolic fixed points $Q = (0^{\mathbb{Z}}, 0)$ expanding and $P = (0^{\mathbb{Z}}, 1/2)$ contracting, and assume that there is $X \in \Sigma_n \times \mathbb{K}$ such that $X \in W^u(P,G) \cap W^s(Q,F)$. After replacing X by some iterate of its orbit, we can assume that

$$X = \left(0^{-\mathbb{N}}.\alpha_0 \cdots \alpha_r 0^{\mathbb{N}}, \frac{1}{2}\right), \tag{1.3.5}$$

for some $(\alpha_0, \dots, \alpha_r)$. This sequence is called the *transition sequence* and the map $g_{[\alpha_0 \cdots \alpha_r]}$ is the *transition map* of the orbit of the heteroclinic point X. We also assume that the transition map $g_{[\alpha_0 \cdots \alpha_r]}$ preserves the orientation in a neighborhood of 1/2.

The map G has a heterodimensional cycle associated to the hyperbolic fixed points P and Q, with heteroclinic intersections

$$\mathcal{I}_{P,Q} := \{0^{\mathbb{Z}}\} \times \left[0, \frac{1}{2}\right]$$

and the orbit of X. The set $\mathcal{I}_{P,Q} \subset W^s(P,G) \cap W^u(Q,G)$ is called *a connection of the cycle.*

A *neighborhood of the cycle* associated to the heteroclinic intersections of the cycle is a open set $V = V(X, \mathcal{I}_{P,Q})$ containing the orbits of the closure of $\mathcal{I}_{P,Q}$ and of X.

We consider a one-parameter family of skew-product maps $(G_t)_{t\in[-1,1]}$ defined by

$$G_t : \Sigma_n \times \mathbb{K} \to \Sigma_n \times \mathbb{K}, \quad G_t(\xi, x) = (\sigma(\xi), g_{\xi_0,t}(x)), t \in [-1,1], \tag{1.3.6}$$

where σ is the shift of n-symbols and, for each $i \in \{0, \dots, n-1\}$,

$$\mathfrak{g}_i : [-1,1] \to C^2(\mathbb{K}, \mathbb{K}), \quad \mathfrak{g}_i(t) = g_{i,t} \tag{1.3.7}$$

is a continuous map such that $g_{i,0} = g_i$. We also assume that $g_{i,t}$ is a C^1-map with respect to the variable t, for all $i \in \{0, \dots, n-1\}$.

Naturally, for $n > 0$, $x \in \mathbb{K}$ and $\xi = (\xi_i)_{i\in\mathbb{Z}} \in \Sigma_n$ we have

$$G_t^n(\xi, x) = (\sigma^n(\xi), g_{[\xi_0 \cdots \xi_{n-1}],t}(x))$$

where $g_{[\xi_0\cdots\xi_{n-1}],t}(x) = g_{\xi_{n-1},t}\circ\cdots\circ g_{\xi_0,t}(x)$.

Now take $\varepsilon > 0$ small. For each $i \in \{0,\dots,n-1\}$, let U_i the set of diffeomorphisms $g_{i,t}$ such that

$$d_{C^2}(g_{i,t}, g_i) < \varepsilon.$$

For each $t > 0$ sufficiently small, let us choose and fix $g_{i,t} \in U_i$. Thus, the mapping $g_{0,t}$ has two fixed points: an attractor p_t close to $1/2$ and a repeller q_t close to 0. The mapping $g_{o,t}$ preserves the orientation on $\mathbb{K}$ and $g'_{0,t}$ is a strictly decreasing map on $[q_t - \iota_t, p_t + \iota_t]$, where $\iota_t > 0$. The points $P_t = (0^{\mathbb{Z}}, p_t)$ and $Q_t = (0^{\mathbb{Z}}, q_t)$ are the *continuations* of $P = (0^{\mathbb{Z}}, 1/2)$ and $Q = (0^{\mathbb{Z}}, 0)$, respectively. We also observe that the mapping $g_{[\alpha_0\cdots\alpha_r],t}$ preserves the orientation in a small neighborhood of p_t.

Definition 1.6. *Assume that $G_0 = G$ has a connected cycle associated to hyperbolic periodic points $P = (0^{\mathbb{Z}}, 1/2)$ contracting and $Q = (0^{\mathbb{Z}}, 0)$ expanding, with heteroclinic intersections $\mathcal{I}_{P,Q} = \{0^{\mathbb{Z}}\} \times [0, 1/2]$ and the orbit of X (see (1.3.5) for the definition of X). Let $g_{[\alpha_0\ \cdots\ \alpha_r]}$ be the transition map associated to the orbit of X. We say that the cycle is* generically unfolded *if*

$$\frac{\partial}{\partial t}\Big(g_{[\alpha_0\cdots\alpha_r],t}(p_t) - q_t\Big)_{t=0} \neq 0.$$

1.3.2 Dynamics in a neighborhood of the cycle

In this subsection, the goal is to define *neighborhood of the cycle*, that is, an open set $\mathcal{V}$ containing all the elements of the cycle, the fixed point P and Q, the intersection $W^s(P,G)\cap W^u(Q,G)$ and the orbit of the heteroclinic point $X = (0^{\mathbb{N}}.\alpha_0\cdots\alpha_r 0^{\mathbb{N}}, 1/2)$, defined in a suitable form.

First, for $k > 0$ and $\epsilon \in (0,\iota)$, we define neighborhood of the connection $\mathcal{I}_{p,q}$ as the set

$$V(\mathcal{I}_{p,q}, k, \epsilon) = \left[0^{-k}.0^k\right] \times \left[-\epsilon, \frac{1}{2} + \epsilon\right], \tag{1.3.8}$$

where 0^k denote the sequence of k consecutive zeros.

Thus, the point Z defined by

$$Z = \left(0^{-\mathbb{N}}.0^k\alpha_0\cdots\alpha_r 0^{\mathbb{N}}, \frac{1}{2}\right) \in W^u(P,G)\cap W^s(Q,G)$$

satisfies $G^{r+1+2k+i}(Z) \in V(\mathcal{I}_{p,q}, k, \epsilon)$ and $G^{-i}(Z) \in V(\mathcal{I}_{p,q}, k, \epsilon)$, for all $i \geq 0$.

Now, we define a neighborhood of the point Z as the set

$$V(Z, k, \gamma) := \left[0^{-k}.0^k\alpha_0 \cdots \alpha_r 0^{2k}\right] \times \left[-\gamma, \frac{1}{2} + \gamma\right] \tag{1.3.9}$$

with $\gamma \in (0, \epsilon)$ such that

$$g_{[0^k\alpha_0\cdots\alpha_r 0^k],0}\left([\frac{1}{2} - \gamma, \frac{1}{2} + \gamma]\right) \subset \left[-\epsilon, \frac{1}{2} + \epsilon\right].$$

Hence, $V(Z, k, \gamma)$ and $G_0^{2k+r+1}(V(Z, k, \gamma))$ are contained in $V(\mathcal{I}_{P,Q}, k, \epsilon)$.

Finally, we define a (k, ϵ, γ)-*neighborhood of the cycle*, $\mathcal{V}(k, \epsilon, \gamma)$, as the set

$$\mathcal{V}(k, \epsilon, \gamma) = V(\mathcal{I}_{p,q}, k, \varepsilon) \bigcup \left(\bigcup_{i=0}^{2k+r} G^i(V(Z, k, \gamma)) \right). \tag{1.3.10}$$

Supposing that the cycle is unfolded for positive t, that is,

$$\frac{\partial}{\partial t}\left(g_{[\alpha_0\cdots\alpha_r],t}(p_t) - q_t\right)_{t=0} > 0, \tag{1.3.11}$$

we have $g_{[\alpha_0\cdots\alpha_r],t}(p_t) \in (q_t, p_t)$. From $g_{0,t}(p_t) = p_t$ and the monotonicity of $g_{0,t}$, if we consider

$$\omega_0 \cdots \omega_{k_0} = 0^k\alpha_0 \cdots \alpha_r 0^k, \;\; k_0 = 2k + r + 1, \tag{1.3.12}$$

we conclude that

$$\frac{\partial}{\partial t}\left(g_{[\omega_0\cdots\omega_{k_0}],t}(p_t) - q_t\right)_{t=0} > 0.$$

Hence, $g_{[\omega_0\cdots\omega_{k_0}],t}$ is the adapted transition map to the neighborhood $\mathcal{V}(k, \epsilon, \gamma)$ of the cycle.

Now, fixed large k and small ϵ and γ, for small $t > 0$, we want to study the *maximal invariant of* G_t in $\mathcal{V}(k, \varepsilon, \gamma)$,

$$\Lambda_t := \Lambda_t(k, \varepsilon, \gamma) = \bigcap_{i\in\mathbb{Z}} G_t^i(\mathcal{V}(k, \varepsilon, \gamma)). \tag{1.3.13}$$

Since k, ϵ and γ are fixed throughout the construction, we write $V(\mathcal{I}_{P,Q})$, $V(Z)$

and $\mathcal{V}$ instead of $V(\mathcal{I}_{P,Q}, k, \epsilon)$, $V(Z, k, \gamma)$ and $\mathcal{V}(k, \epsilon, \gamma)$, respectively.

1.3.3 Systems of iterated functions

In this section the goal is to construct a system of iterated functions $\mathfrak{G}_{d,t}$ generated by $g_{i,t}$, $i \in \{1, \ldots, n-1\}$ describing the dynamics of G_t in the neighborhood $\mathcal{V}$ of the cycle.

We begin by choosing $t_0 > 0$ and taking $A_{t_0} = \left(0^{-\mathbb{N}}.\omega_0 \cdots \omega_{k_0} 0^{\mathbb{N}}, p_{t_0}\right) \in W^u(P, G_{t_0})$. Since $g_{[\omega_0 \cdots \omega_{k_0}],0}(1/2) = 0$, from (1.3.11), we conclude that there are

$$d_{t_0} \in \left(g_{[\omega_0 \cdots \omega_{k_0}],t_0}(p_{t_0}), p_{t_0}\right)$$

and $h \in \mathbb{N}$ such that $g_{[0^h \omega_0 \cdots \omega_{k_0}],t_0}(d_{t_0}) = q_{t_0}$, and, once each $\mathfrak{g}_i$ is a continuous map, for t close to t_0, there is d_t such that $g_{[0^h \omega_0 \cdots \omega_{k_0}],t}(d_t) = q_t$.

In (q_t, p_t), we consider $D_t := [d_t, g_{0,t}(d_t)]$ a fundamental domain of $g_{0,t}$ and we define the "cube" Δ_t by

$$\Delta_t := \left[0^{-k}.0^k\right] \times D_t \tag{1.3.14}$$

which is contained in $\mathcal{V}$. In what follows we consider returns by G_t of points in the cube Δ_t to itself.

Definition 1.7. *Given $X = X_0 \in \Delta_t$, we define the sequence of return times $(\varrho_i(X))_{i \in I(X)}$ of X to Δ_t by*

- $\varrho_0(X) = 0$
- $\varrho_1(X_i) = X_{i+1}$
- $\varrho_i(X) < \varrho_{i+1}(X)$, $G_t^{\varrho_i(X)}(X) \in \Delta_t$ *for all* $i \in I(X)$, *and* $G_t^j(X) \notin \Delta_t$ *for each* $\varrho_i(X) < j < \varrho_{i+1}(X)$,

where $I(X)$ is a (maximal) interval of $\mathbb{Z}$ containing 0 (this interval may be upper or/and lower bound). We denote by $X_{[i]}$ the i-th return of $X = X_0$.

The following lemma is a adaptation to the context of skew-product maps of the Lemma 7.1 in [DR02], proved for heterodimensional cycles through the existence of the filtration and the geometry of the cycle. The proof of this result is done using similar arguments.

Lemma 1.1. ([DR02, Lemma 7.1]) *Consider small $t > 0$ and $X \in \Delta_t \cap \Lambda_t$. There are three possibilities:*

1. *X has infinitely many forward and backward returns $\varrho_i(X)$,*
2. *$X \in W^s(P, G_t) \cup W^s(Q, G_t)$, if and only if, X has only finitely forward returns $\varrho_i(X)$, $i > 0$, and*
3. *$X \in W^u(P, G_t) \cup W^u(Q, G_t)$, if and only if, X has only finitely backward returns $\varrho_i(X)$, $i < 0$.*

Consider a point $X \in \Delta_t \cap \Lambda_t$ such that $X \notin W^s(P, G_t) \cup W^s(Q, G_t)$. Then after $\varrho_1(X)$ positive iterations, $\varrho_1(X) = u_1 + k_0 + s_1$, where k_0 is as in (1.3.12), this point has a return to Δ_t. Thus $X = (\cdots .0^{s_1}\omega_0 \cdots \omega_{k_0} 0^{u_1} \cdots, x)$ and:

- $G_t^{s_1}(X) \in V(Z)$,
- $G_t^{k_0+s_1}(X) \in V(\mathcal{I}_{P,Q})$,
- $G_t^{i+k_0+s_1}(X) \in (V(\mathcal{I}_{P,Q}) \setminus \Delta_t)$ for all $0 \leq i < u_1$, and
- $G_t^{u_1+k_0+s_1}(X) \in \Delta_t$.

We say that the point X has a *return of type* (u_1, s_1).

If $X_{[1]}, \ldots, X_{[m]}$ are m consecutive returns of $X = X_{[0]}$, where $X_{[i]}$ is a return of $X_{[i-1]}$ of type (u_i, s_i), then we say that $X_{[m]}$ is the *m-th return of $X_{[0]}$ of type* $\mathfrak{b}_m = (u_1, s_1) \cdots (u_m, s_m)$. If X is a periodic point of G_t and m is the smallest positive integer such that $X_{[m]} = X$, then $\mathfrak{b}_m$ is called the *periodic itinerary of* X.

Let $X_{[i]} = (\eta, x_{[i]})$ the i-th return of a point $X \in \Delta_t$. Thus, from the definition of G_t, the coordinate $x_{[i+1]}$ of $X_{[i+1]}$ is

$$x_{[i+1]} = g_{[0^{s_i}\omega_0 \cdots \omega_{k_0} 0^{u_i}],t}(x_{[i]}) = g_{0,t}^{u_i} \circ g_{[\omega_0 \cdots \omega_{k_0}],t} \circ g_{0,t}^{s_i}(x_{[i]}).$$

This leads us to define, for each $(u, s) \in \mathbb{N}_0^2$, the following one-parameter family of maps by

$$\Gamma_{d,t}^{(u,s)} : D_t^{(u,s)} \to D_t, \qquad \Gamma_{d,t}^{(u,s)}(x) \;=\; g_{0,t}^{u} \circ g_{[\omega_0 \cdots \omega_{k_0}],t} \circ g_{0,t}^{s}(x) \qquad (1.3.15)$$

where $D_t^{(u,s)}$ is the maximal subinterval of D_t where $\Gamma_{d,t}^{(u,s)}$ is defined. Note that, there exist pairs (u,s) for which $D_t^{(u,s)}$ is a empty set. Given a *chain of pairs* $\mathfrak{b} = \mathfrak{b}_k = (u_1,s_1)\cdots(u_k,s_k)$, with $u_i, s_i \in \mathbb{N}_0$, in similar way, we can define the map

$$\Gamma_{d,t}^{\mathfrak{b}} : D_t^{\mathfrak{b}} \to D_t, \quad \Gamma_{d,t}^{\mathfrak{b}}(x) = \Gamma_{d,t}^{(u_k,s_k)} \circ \cdots \circ \Gamma_{d,t}^{(u_1,s_1)}(x), \tag{1.3.16}$$

where $D_t^{\mathfrak{b}} \subset D_t$ is the maximal domain of definition of $\Gamma_{d,t}^{\mathfrak{b}}$. We observe that, since the maps $\Gamma_{d,t}^{\mathfrak{b}}$ are compositions of $g_{0,t}$ and $g_{[\alpha_0,\dots,\alpha_r],t}$, which preserve the orientation in $\mathbb{K}$ and in a neighborhood of p_t, respectively, the maps $\Gamma_{d,t}^{\mathfrak{b}}$ are also order preserving.

We now define the one-parameter family of iterated function systems, $\mathfrak{G}_{d,t}$, associated to the cycle and the cube Δ_t

$$\mathfrak{G}_{d,t} := \{\Gamma_{d,t}^{\mathfrak{b}} : \mathfrak{b} \text{ is a chain of length } k \in \mathbb{N}\ \} \tag{1.3.17}$$

For simplicity, in the notation we drop d.

Notation 1.1 To each chain $\mathfrak{b} = (u_1,s_1)\cdots(u_k,s_k)$ we associate the sequence

$$\theta(\mathfrak{b}) := 0^{s_1}\omega_0\cdots\omega_{k_0}0^{u_1}\cdots 0^{s_k}\omega_0\cdots\omega_{k_0}0^{u_k} \tag{1.3.18}$$

thus, we have $\Gamma_t^{\mathfrak{b}}(x) = g_{[\theta(\mathfrak{b})],t}(x)$ and the *length* of $\theta(\mathfrak{b})$ is defined as

$$|\theta(\mathfrak{b})| = \sum_{i=1}^{k}(s_i + k_0 + 1 + u_i).$$

At last, we introduce a further definition. Given two chains $\mathfrak{b} = (u_1,s_1)\cdots(u_k,s_k)$ and $\overline{\mathfrak{b}} = (\overline{u}_1,\overline{s}_1)\cdots(\overline{u}_m,\overline{s}_m)$ its *composition* is defined by

$$\mathfrak{b} * \overline{\mathfrak{b}} = (u_1,s_1)\cdots(u_k,s_k)(\overline{u}_1,\overline{s}_1)\cdots(\overline{u}_m,\overline{s}_m).$$

Chapter 2

Two model families for non-hyperbolic homoclinic classes

In this chapter, for each $a > 0$, we present a one-parameter family of skew-product maps, $(G_{a,t})_{t\in[-1,1]}$, unfolding a heterodimensional cycle at $t = 0$ associated to two hyperbolic fixed points P and Q.

First, we consider $0 < a < \log 2$ and the goal is to prove that, after the unfolding and for every $t > 0$ small, the relative homoclinic classes of P and Q to $\mathcal{V}$, where $\mathcal{V}$ is a neighborhood of the cycle (see (1.3.10)), explode and become equal.

For $\log 2 < a < \log 4$ and for a subset of the parameter space we prove that $H_{\mathcal{V}}(Q, G_{a,t}) = H_{\mathcal{V}}(P, G_{a,t})$.

We also present a model family of diffeomorphisms, $(f_{a,t})_{t\in[-\tau,\tau]}$ unfolding a heterodimensional cycle at $t = 0$ associated to the fixed saddles $P = (0, 1/2, 0)$ and $Q = (0, 0, 0)$, and we derive similar conclusions for this family as we done for $G_{a,t}$.

2.1 Skew-product maps: a model family

In this section, for each $a > 0$, we construct the arc of skew-product maps $(G_{a,t})_{t\in[-1,1]}$ and state the main results.

Unless otherwise stated, we assume that $n = 2$. Consider a two-parameter family of skew-product maps $G_{a,t} : \Sigma_2 \times (-1/(2(e^a - 1)), 1] \to \Sigma_2 \times \mathbb{R}$, with $t \in [-1, 1]$ and $a > 0$, such that $g_{0,t} = g_a$ is defined as the time one of the vector field $x' = -2ax(1 - 2x)$. The map g_a has two fixed points in $(-1/(2(e^a - 1)), 1]$, 0 and 1/2, and, for every $x \in [0, 1/2]$ and $n \in \mathbb{Z}$, we have

$$g_a^n(x) = \frac{xe^{na}}{2xe^{na} + (1-2x)}, \tag{2.1.1}$$

which, naturally, is a differentiable function with

$$(g_a^n)'(x) = \frac{e^{-na}}{x^2} \left(g_a^n(x)\right)^2, \ x \neq 0, \quad \text{and } (g_a^n)'(0) = e^{na}. \tag{2.1.2}$$

In particular, we have $g_a'(0) = e^a$ and $g_a'(1/2) = e^{-a}$.

Let $g_{1,0} = g_1$ (the map is independent of a parameter) be the transition map such that g_1 is strictly increasing and satisfies the requirements

$$\frac{\partial}{\partial t}\left(g_{1,t}\left(\frac{1}{2}\right)\right)_{t=0} > 0 \text{ and } g_{1,0}\left(\frac{1}{2}\right) = 0.$$

Consequently we may define a differentiable map $\mathfrak{g}_1$ (see (1.3.7)), for $i = 1$, such that $g_{1,t}$ is affine in a neighborhood of $1/2$, that is, for $\zeta > 0$ small enough, there are $b,\ c > 0$ such that

$$g_{1,t}(x) = b(x - 1/2) + ct, \text{ for all } x \in \left[\frac{1}{2} - \zeta, \frac{1}{2} + \zeta\right]. \tag{2.1.3}$$

Therefore $Y_t = (0^{-\mathbb{N}}.10^{\mathbb{N}}, 1/2 - ct/b)$ is a transverse homoclinic point of $Q = (0^{\mathbb{Z}}, 0)$ and $X_t = (0^{-\mathbb{N}}1.0^{\mathbb{N}}, ct)$ is a transverse homoclinic point of $P = (0^{\mathbb{Z}}, 1/2)$. Without loss of generality, by a convenient choice of the fundamental domain of g_a, we may assume that $b = c = 1$.

Fix $k \in \mathbb{N}$. Since the transition map is given by one map $g_{1,t}$, we can consider the neighborhood of the cycle (1.3.10) with

$$V(Z, k, \gamma) = \left[0^{-k}.0^k 10^{2k}\right] \times \left[\frac{1}{2} - \gamma, \frac{1}{2} + \gamma\right]$$

instead of (1.3.9) and $\gamma < \zeta$. Thus the adapted transition map is given by

$$g_{[0^k 10^k],t} = g_{0,t}^k \circ g_{1,t} \circ g_{0,t}^k,$$

that is $\omega_0 \dots \omega_{k_0} = 0^k 10^k$.

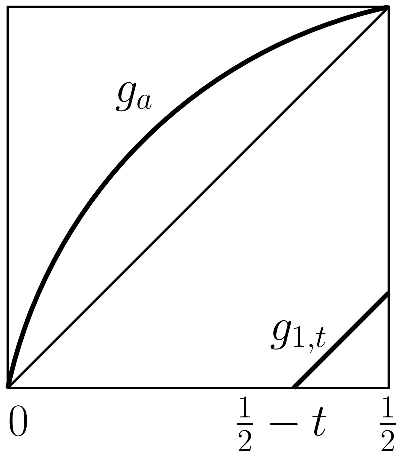

Figure 2.1: The diffeomorphisms g_a and $g_{1,t}$

Consider the set

$$T = \{t > 0 : W^u(P, G_{a,t}) \cap W^s(Q, G_{a,t}) \neq \emptyset\}. \tag{2.1.4}$$

We observe that, for the parameters $t \in T$, one has $W^u(P, G_{a,t}) \cap W^s(Q, G_{a,t}) \neq \emptyset$ and, by construction, $\{0\}^{\mathbb{Z}} \times (0, 1/2) \subseteq W^u(Q, G_{a,t}) \cap W^s(P, G_{a,t})$. Consequently $G_{a,t}$ has a heterodimensional cycle associated to P and Q. Thus, the parameters $t \in T$ correspond to secondary cycles. In Section 2.4, we will see that $H_{\mathcal{V}}(Q, G_{a,t})$ and $W^u(P, G_{a,t}) \cap W^s(Q, G_{a,t})$ are disjoint sets (see (2.4.3)).

Now, we state one of the main results of this thesis whose proof is presented in Section 2.2.

Theorem 2.1. *For each* $0 < a < \log 2$ *there is* $t_0 = t_0(a) > 0$ *such that*

$$H_{\mathcal{V}}(P, G_{a,t}) \subseteq H_{\mathcal{V}}(Q, G_{a,t}), \ \forall t \in (0, t_0].$$

Moreover, if $t \notin T$ *then* $\Lambda_{a,t} = H_{\mathcal{V}}(Q, G_{a,t})$.

In spite of the proof will be presented in the following section, it is important to say that the key step of it is to prove that, for each $0 < a < \log 2$ and x in a fundamental domain of g_a, there are a constant $l_a > 1$, independent of small $t \in (0, t_0]$, and a chain $\mathfrak{b} = \mathfrak{b}(x) = (u_1, s_1) \cdots (u_k, s_k), \ k \in \{1, 2\}, \ u_i, s_i \in \mathbb{N}_0$ such that

$$\left(\Gamma^{\mathfrak{b}}_{a,t}\right)'(x) > l_a > 1, \tag{2.1.5}$$

where $\Gamma^{\mathfrak{b}}_{a,t}$ is a function of the form (1.3.16) (see Proposition 2.5). From this we deduce that, for every interval $I \subset (0, 1/2)$, there are $(\vartheta_0, \dots, \vartheta_m)$, $\vartheta_i \in \{0, 1\}$ and a point $x^\star \in I$ such that $g_{[\vartheta_0 \cdots \vartheta_m],t}(x^\star) = 0$ (see Corollary 2.6 and Proposition 2.7). This fact allows us to prove that, for $t \notin T$, the relative homoclinic class of Q to $\mathcal{V}$ is equal to the the maximal invariant of $G_{a,t}$ in $\mathcal{V}$, $\Lambda_{a,t} = \bigcap_{i \in \mathbb{Z}} G^i_{a,t}(\mathcal{V})$.

Note that, for $a = \log 2$, we can prove the condition (2.1.5) but with $l = l_t > 1$ depending on t, that is, $l_t \to 1$ as t goes to zero. Since t is fixed along the proof of Theorem 2.1, the same conclusion holds for $a = \log 2$ and $0 < t < t_0(a)$. We observe that, from Lemma 2.4, the size of $t_0 = t_0(a)$ decreases as a increases and

$$\lim_{a \to \log 2} t_0(a) = 0.$$

For this special family, by symmetric properties of g_a (see Proposition 2.9) and interchanging the roles of P and Q, we can apply Theorem 2.1 to $G^{-1}_{a,t}$ to obtain

$$H_{\mathcal{V}}(Q, G^{-1}_{a,t}) \subset H_{\mathcal{V}}(P, G^{-1}_{a,t}) \text{ which means } H_{\mathcal{V}}(Q, G_{a,t}) \subset H_{\mathcal{V}}(P, G_{a,t}).$$

Corollary 2.2. *For $0 < a < \log 2$ there is $t_0 = t_0(a)$ such that, for all $t \in (0, t_0]$, $H_{\mathcal{V}}(P, G_{a,t}) = H_{\mathcal{V}}(Q, G_{a,t})$.*

For $\log 2 < a < \log 4$, we will prove that there exists a set of parameters t such that the homoclinic classes of P and Q are equal, which prevents the hyperbolicity. The main difference between this situation and the previous one, when $0 < a < \log 2$, is the following: while here we can find a sequence $t_n \to 0$, as $n \to \infty$, such that $D^{(1,n-2k)}_{a,t_n} \neq \emptyset$ for n large, for $a \in (0, \log 2]$ we have $D^{(1,n-2k)}_{a,t} = \emptyset$ for all t close to zero, where $D^{(1,n-2k)}_{a,t}$ is the maximal subinterval of $D_{a,t}$ where $\Gamma^{(1,n-2k)}_{a,t}$ is defined (see (1.3.16)).

Theorem 2.3. *For $a \in (\log 2, \log 4)$ there are $t_0(a) > 0$, a sequence $t_n(a) \in (0, t_0(a)]$ converging to zero as $n \to +\infty$, and a sequence of intervals*

$$J(a, t_n) = [t_n(a) - \alpha_{a,t_n}, \alpha_{a,t_n} + t_n(a)]$$

such that $H_{\mathcal{V}}(P, G_{a,t}) = H_{\mathcal{V}}(Q, G_{a,t})$ for $t \in J(a, t_n)$, and $\Lambda_{a,t} = H_{\mathcal{V}}(Q, G_{a,t})$ for $t \in J(a, t_n) \setminus T$.

As we will see in Section 2.2, $t_n = t_n(a)$ is defined by the equation $g_a^n(t_n) = 1/2 - t_n$, thus the point

$$X = (0^{-\mathbb{N}}1.0^n10^{\mathbb{N}}, t_n) \in W^s(Q, G_{a,t_n}) \cap W^u(P, G_{a,t_n})$$

and consequently the sequence $(t_n)_n$ corresponds to secondary cycles.

To prove this result we follow several steps. First, we find $n_0 \in \mathbb{N}$ large enough such that $D_{a,t_n}^{(1,n-2k)} \neq \emptyset$ and $\Gamma_{a,t_n}^{(1,n-2k)}$ has no fixed points for all $n \geq n_0$. Then, we show that the system $\mathfrak{G}_{a,t_n}$ satisfies an expanding property, and, arguing as in Theorem 2.1, we conclude that $H_{\mathcal{V}}(P, G_{a,t_n}) = H_{\mathcal{V}}(Q, G_{a,t_n})$.

2.2 Explosion of the relative homoclinic classes

As we said above, in this section we prove Theorem 2.1. We begin by studying the dynamics of the system of iterated functions $\mathfrak{G}_{a,t}$ for $a \in (0, \log 2)$ and t small.

Let $a \in (0, \log 2)$. For each $n \in \mathbb{N}$ large, we can choose $t_n = t_n(a) > 0$ small enough such that $t_n(a) < \zeta$ (see (2.1.3)) and

$$g_a^n(t_n(a)) = 1/2 - t_n(a). \tag{2.2.1}$$

Naturally, the dependence on a of the sequence $(t_n(a))_n$ is present but, for simplicity of notation, in what follows we are going to write $(t_n)_n$ instead of $(t_n(a))_n$. From the definition of $(t_n)_n$, we have

$$e^{-na} = \frac{(2t_n)^2}{(1-2t_n)^2}, \tag{2.2.2}$$

hence $t_n \to 0$ as $n \to +\infty$ and

$$\lim_{n\to+\infty} \frac{t_{n+1}}{t_n} = \lim_{n\to\infty} \left(\frac{1}{(1-2t_n)e^{\frac{a}{2}} + 2t_n} \right) = e^{-\frac{a}{2}} \tag{2.2.3}$$

So, the intervals $(t_{n+1}, t_n]$ have positive density at zero.

Now, fix t small enough. Then there is n sufficiently large such that $t \in (t_{n+1}, t_n]$, consequently $t = t_n(1+\mu)$ with $\mu = \mu(t_n, a) \leq 0$ (for simplicity we ignore the dependence of t_n and of a on μ).

It is not difficult to see that $(t_{n+1} - t_n)/t_n$ increases with t_n and

$$\lim_{n\to+\infty} \frac{t_{n+1} - t_n}{t_n} = e^{-\frac{a}{2}} - 1, \tag{2.2.4}$$

thus we have that $e^{-\frac{a}{2}} < 1 + \mu \leq 1$ which gives the interval of variation of μ. For each n, we take $t_{n+1}/t_n < 1 + \mu \leq 1$.

Using (2.2.2), we can write (2.1.1) and (2.1.2) with the form

$$g_a^n(x) = \frac{x(1-2t_n)^2}{2x(1-2t_n)^2 + (1-2x)(2t_n)^2} \tag{2.2.5}$$

and

$$(g_a^n)'(x) = \left(\frac{2t_n(1-2t_n)}{2x(1-2t_n)^2 + (1-2x)(2t_n)^2}\right)^2, \tag{2.2.6}$$

respectively.

For fixed $t \in (t_{n+1}, t_n]$, $t = t_n(1+\mu)$, we consider the fundamental domain of g_a

$$D_{a,t} := \left[g_a^k(d_{a,t}), g_a^{k+1}(d_{a,t})\right], \text{ where } g_a^n(d_{a,t}) = \frac{1}{2} - t,$$

and k is defined in (1.3.8). Thus $g_{1,t} \circ g_a^{n-k}(g_a^k(d_{a,t})) = 0$ and

$$d_{a,t} = \frac{(1-2t_n(1+\mu))t_n}{2(1-2t_n(1+\mu))t_n + (1+\mu)(1-2t_n)^2}. \tag{2.2.7}$$

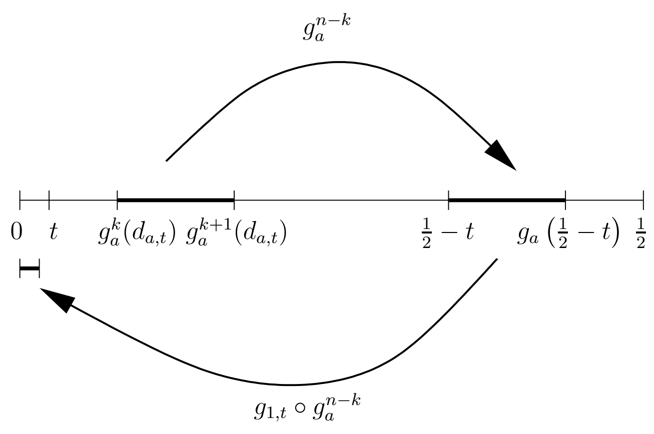

Figure 2.2: A return map to $D_{a,t}$

We claim that

$$g_a^{k+1}(t) \in D_{a,t}.$$

In fact, since $t \in (t_{n+1}, t_n]$ and $g_a^{n+1}(t_{n+1}) = 1/2 - t_{n+1}$, then

$$\begin{aligned} g_a^{n+1}(t) \in (g_a^{n+1}(t_{n+1}), g_a^{n+1}(t_n)] &= \left(\tfrac{1}{2} - t_{n+1}, g_a\left(\tfrac{1}{2} - t_n\right)\right] \\ &\subseteq \left[\tfrac{1}{2} - t, g_a\left(\tfrac{1}{2} - t\right)\right] \\ &= [g_a^n(d_{a,t}), g_a^{n+1}(d_{a,t})], \end{aligned}$$

hence $d_{a,t} < g_a(t) < g_a(d_{a,t})$, that is $g_a^{k+1}(t) \in D_{a,t}$.

For small $t > 0$ and $u \in \mathbb{N}_0$ consider the maps

$$\begin{aligned} \Gamma_{a,t}^{(u,n-2k)} : \quad D_{a,t}^{(u,n-2k)} &\to D_{a,t} \\ x &\mapsto g_{[0^{n-k}10^{u+k}],t}(x) = g_a^{u+k} \circ g_{1,t} \circ g_a^{n-k}(x), \end{aligned}$$

where $D_{a,t}^{(u,n-2k)} := \{x \in D_{a,t} : \Gamma_{a,t}^{(u,n-2k)}(x) \in D_{a,t}\}$. The next lemma implies that, for each $a \in (0, \log 2)$, there is a positive small number $t_0 = t_0(a)$ such that, for every $t \in (0, t_0]$, we get $D_{a,t}^{(1,n-2k)} = \emptyset$. Thus, we also have $D_{a,t}^{(0,n-k)} = \emptyset$.

Lemma 2.4. *For every $a \in (0, \log 2)$ there is small $t_0 = t_0(a) > 0$ such that, for every $t \in (0, t_0]$,*

$$g_a^{k+1} \circ g_{1,t} \circ g_a^{n-k}(x) < g_a^k(d_{a,t}), \text{ for all } x \in D_{a,t}.$$

Proof. It follows from the monotonicity of g_a and $g_{1,t}$ that, for all $x \in D_{a,t}$,

$$g_a^{k+1} \circ g_{1,t} \circ g_a^{n-k}(x) < g_a^{k+1} \circ g_{1,t} \circ g_a^{n-k}\left(g_a^{k+1}(d_{a,t})\right),$$

so it is sufficient to prove that

$$g_a^{k+1} \circ g_{1,t} \circ g_a^{n-k}\left(g_a^{k+1}(d_{a,t})\right) < g_a^k(d_{a,t}),$$

that is, $g_{1,t} \circ g_a^{n-k}(g_a^{k+1}(d_{a,t})) < g_a^{-1}(d_{a,t})$ or equivalently $g_{1,t} \circ g_a^n(g_a(d_{a,t})) < g_a^{-1}(d_{a,t})$.

On the other hand, since g_a^{-1} is an increasing map and $g_a^{k+1}(t) \in D_{a,t}$, we also

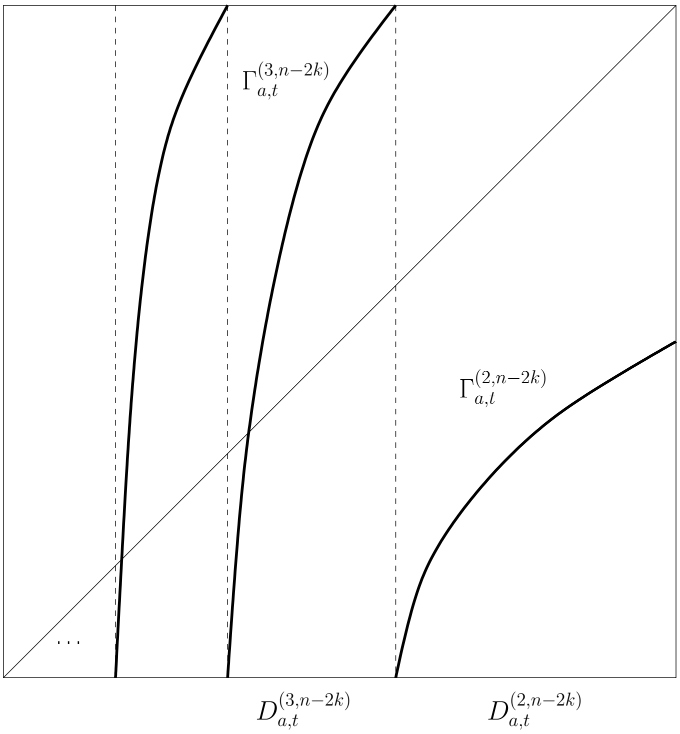

Figure 2.3: The maps $\Gamma^{(u,n-2k)}_{a,t}$

have $g_a^{-1}(t) \leq g_a^{-1}(d_{a,t})$. Thus we shall have established the lemma if we check that

$$g_{1,t} \circ g_a^n(g_a(d_{a,t})) < g_a^{-1}(t).$$

By definition of $d_{a,t}$ we have $g_{1,t} \circ g_a^n(g_a(d_{a,t})) = g_{1,t} \circ g_a \circ g_a^n(d_{a,t}) = g_{1,t} \circ g_a\,(1/2 - t)$ and

$$\begin{aligned} g_{1,t} \circ g_a\left(\tfrac{1}{2} - t\right) &= \frac{(1-2t)e^a}{2(1-2t)e^a + 4t} - \frac{1}{2} + t \\ &= \frac{t(-2 + 2(1-2t)e^a + 4t)}{2(1-2t)e^a + 4t}. \end{aligned}$$

Now writing $t = t_n(1+\mu)$ and since $t_n \to 0$ as $n \to +\infty$, by some calculations it follows

$$\lim_{n\to+\infty} \frac{g_{1,t} \circ g_a^n(g_a(d_{a,t})) - g_a^{-1}(t_n(1+\mu))}{t_n} = (1+\mu)(-2+e^a)e^{-a} < 0.$$

Therefore, for every $a \in (0, \log 2)$, there is small $t_0 = t_0(a)$ (the size of $t_0(a)$ goes to zero as a goes to $\log 2$) such that, for all $t \in (0, t_0]$ one has $g_{1,t} \circ g_a^n(g_a(d_{a,t})) < t \leq d_{a,t}$, which completes the proof. □

The main technical result of this section is the following:

Proposition 2.5. *There is $l_a > 1$ such that, for every $t \in (0, t_0]$ and every $x \in D_{a,t}$, there is a chain $\mathfrak{b} = (u_1, n-2k) \cdots (u_k, n-2k)$, $k = 1$ or 2 such that*

$$(\Gamma_{a,t}^{\mathfrak{b}})'(x) \geq l_a.$$

Moreover $l_a \to 1$ when $a \to \log 2$.

A straightforward consequence of this proposition, which the proof is done below, is the following result. In the sequel, for an interval I, $|I|$ denotes its length and, given k chains $\mathfrak{b}_1, \cdots, \mathfrak{b}_k$, we denote by $\mathfrak{b}_i^\star$ the chain $\mathfrak{b}_1 * \cdots * \mathfrak{b}_i$, for all $i \in \{1, \cdots, k\}$.

Corollary 2.6. *For every $t \in (0, t_0]$ and every interval $I \subseteq (g_a^k(d_{a,t}), g_a^{k+1}(d_{a,t})]$, there is a chain $\mathfrak{b}^\star = \mathfrak{b}_1 * \mathfrak{b}_2 \cdots * \mathfrak{b}_k$ such that $\Gamma_{a,t}^{\mathfrak{b}_i^\star}(I) \subset (g_a^k(d_{a,t}), g_a^{k+1}(d_{a,t})]$, for all $i \in \{0, 1, \ldots, k-1\}$, and the interval $\Gamma_{a,t}^{\mathfrak{b}^\star}(I)$ satisfies*

$$|\Gamma_{a,t}^{\mathfrak{b}^\star}(I)| \geq l_a^k |I|.$$

Therefore, for every interval $I \subset (g_a^k(d_{a,t}), g_a^{k+1}(d_{a,t})]$, there is a chain $\mathfrak{b}$ such that $g_a^k(d_{a,t}) \in \Gamma_{a,t}^{\mathfrak{b}}(I)$.

Proof of Proposition 2.5. Given $t \in (0, t_0]$, where t_0 is defined in the previous lemma, there is n large such that $t \in (t_{n+1}, t_n]$. First note that $D_{a,t} = \bigcup_{u \in \mathbb{N}} D_{a,t}^{(u,n-2k)}$ and, by the previous lemma, for $0 < a < \log 2$ we have $u \geq 2$. The proof is divided in two steps.

Claim 1. For $u_1 \geq 3$, we have

$$\left(\Gamma_{a,t}^{(u_1,n-2k)}\right)'(x) > e^{(u_1-2)a}(1+\mu)^2 = A > 1, \tag{2.2.8}$$

for all $x \in D_{a,t}^{(u_1,n-2k)}$.

In fact, given $x \in D_{a,t}^{(u_1,n-2k)}$, we obtain

$$\lim_{n \to +\infty} \left(\Gamma_{a,t}^{(u_1,n-2k)}\right)'(x) = \lim_{n \to +\infty} \left(g_a^{u_1+k} \circ g_{1,t} \circ g_a^{n-k}\right)'(x),$$

thus

$$\begin{aligned}\lim_{n\to+\infty}\left(\Gamma_{a,t}^{(u_1,n-2k)}\right)'(x) &\geq \lim_{n\to+\infty}\left(g_a^{u_1+k}\circ g_{1,t}\circ g_a^{n-k}\right)'\left(g_a^{k+1}(d_{a,t})\right)\\ &= e^{(u_1+k)a}\lim_{n\to+\infty}(g_a^{n-k})'(g_a^{k+1}(d_{a,t})). \qquad (2.2.9)\end{aligned}$$

where the last inequality follows from the fact that $(g_a^{u_1+k})'(0) = e^{(u_1+k)a}$. From equations (2.1.2) and (2.2.6), it follows

$$\begin{aligned}(g_a^{n-k})'(g_a^{k+1}(d_{a,t})) &= e^{ka}\frac{(2t_n)^2}{(1-2t_n)^2}\left(\frac{g_a^{n+1}(d_{a,t})}{g_a^{k+1}(d_{a,t})}\right)^2\\ &= e^{ka}\frac{(2t_n)^2}{(1-2t_n)^2}\left(\frac{g_a^{n+1}(d_{a,t})}{d_{a,t}}\right)^2\left(\frac{d_{a,t}}{g_a^{k+1}(d_{a,t})}\right)^2,\end{aligned}$$

and from (2.1.1), (2.2.5), and (2.2.7) we obtain

$$\begin{aligned}&\lim_{n\to\infty}(g_a^{n-k})'(g_a^{k+1}(d_{a,t})) =\\ &= e^{ka}\times\lim_{n\to\infty}\frac{4t_n^2}{(1-2t_n)^2}\frac{(1-2t_n)^4}{(2d_{a,t}(1-2t_n)^2+(1-d_{a,t})4t_n^2e^{-a})^2}\left(\frac{d_{a,t}}{g_a^{k+1}(d_{a,t})}\right)^2\\ &= \frac{e^{ka}}{(e^{(k+1)a})^2}\lim_{n\to+\infty}\frac{(1-2t_n)^2}{\left(\dfrac{d_{a,t}}{t_n}(1-2t_n)^2+(1-2d_{a,t})2t_ne^{-a}\right)^2}\\ &= e^{-(k+2)a}(1+\mu)^2. \qquad (2.2.10)\end{aligned}$$

Thus from (2.2.9) and (2.2.10) we conclude that

$$\lim_{n\to+\infty}\left(\Gamma_{a,t}^{(u_1,n-2k)}\right)'(x) \geq e^{(u_1-2)a}(1+\mu)^2.$$

Since $u_1 \geq 3$, from (2.2.4) $e^{-\frac{a}{2}} < 1+\mu \leq 1$ it follows $e^{(u_1-2)a}(1+\mu)^2 > 1$ and Claim 1 is proved.

Now consider $u_1 = 2$ and $x \in D_{a,t}^{(2,n-2k)}$. We shall have established the proposition if we prove the following:

Claim 2 For $u_1 = 2$ and $x \in D_{a,t}^{(2,n-2k)}$, there is $u_2 \geq 2$ such that

$$x \in D_{a,t}^{\mathfrak{b}} \text{ and } (\Gamma_{a,t}^{\mathfrak{b}})'(x) > \frac{e^{u_2-2}}{(e^a-1)^2} = B > 1, \tag{2.2.11}$$

where $\mathfrak{b} = (2, n-2k)(u_2, n-2k)$.

From the monotonicity of g_a and g_a' it is enough to show that

$$\lim_{n\to+\infty} (\Gamma_{a,t}^{\overline{\mathfrak{b}}})'(g_a^{k+1}(d_{a,t})) > 1,$$

where $\overline{\mathfrak{b}} = (2, n-2k)(\overline{u_2}, n-2k)$ such that $g_a^{k+1}(d_{a,t}) \in D_{a,t}^{\overline{\mathfrak{b}}}$, with $2 \leq \overline{u_2} \leq u_2$.

Once in Claim 1 we only use the hypothesis $u_1 \geq 3$ in the last inequality, from (2.2.11) we have

$$\lim_{n\to\infty} (\Gamma_{a,t}^{(2,n-2k)})'(g_a^{k+1}(d_{a,t})) = (1+\mu)^2.$$

Recalling that $g_a^n(d_{a,t}) = 1/2 - t_n(1+\mu)$, it follows

$$\begin{aligned}
\lim_{n\to+\infty} \frac{\Gamma_{a,t}^{(2,n-2k)}\left(g_a^{k+1}(d_{a,t})\right)}{t_n} &= \lim_{n\to\infty} \frac{g_a^{k+2} \circ g_{1,t} \circ g_a\left(\frac{1}{2} - t_n(1+\mu)\right)}{t_n} \\
&= e^{(k+1)a}(e^a-1)(1+\mu)
\end{aligned}$$

From this, and writing $\overline{d_{a,t}} := \Gamma_{a,t}^{(2,n-2k)}\left(g_a^{k+1}(d_{a,t})\right)$ for the sake of simplicity of notation, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \left(\Gamma_{a,t}^{(\overline{u_2},n-2k)}\right)'\left(\overline{d_{a,t}}\right) &= \lim_{n\to+\infty} \left(g_a^{\overline{u_2}+k} \circ g_{1,t} \circ g_a^{n-k}\right)'\left(\overline{d_{a,t}}\right) \\
&= e^{(\overline{u_2}+k)a} \lim_{n\to+\infty} \left(g_a^{n-k}\right)'\left(\overline{d_{a,t}}\right) \\
&= e^{(\overline{u_2}+k+k)a} \lim_{n\to+\infty} \left(\frac{(1-2t_n)}{\frac{\overline{d_{a,t}}}{t_n}(1-2t_n)^2 + (1-2\overline{d_{a,t}})2t_n e^{ka}}\right)^2 \\
&= e^{(\overline{u_2}+2k)a} \left(\frac{1}{e^{(k+1)a}(e^a-1)(1+\mu)}\right)^2,
\end{aligned}$$

therefore

$$\lim_{n\to\infty}\left(\Gamma_{a,t}^{(\overline{u_2},n-2k)}\right)'\left(\overline{d_{a,t}}\right)=\frac{e^{(\overline{u_2}-2)a}}{(e^a-1)^2(1+\mu)^2}$$

Putting together these facts, for $x\in D_{a,t}^{(2,n-2k)}$, it is trivial to conclude that there is $u_2\geq\overline{u_2}$ such that, for $\mathfrak{b}=(2,n-2k)(u_2,n-2k)$, we have the following straightforward estimates

$$\begin{aligned}\lim_{n\to+\infty}(\Gamma_{a,t}^{\mathfrak{b}})'(x) &\geq \lim_{n\to+\infty}(\Gamma_{a,t}^{\overline{\mathfrak{b}}})'(g_a^{k+1}(d_{a,t}))\\ &= \lim_{n\to+\infty}\left(\Gamma_{a,t}^{(\overline{u_2},n-2k)}\right)'\left(\overline{d_{a,t}}\right)\left(\Gamma_{a,t}^{(2,n-2k)}\right)'\left(g_a^{k+1}(d_{a,t})\right)\\ &= \frac{e^{(\overline{u_2}-2)a}\left(1+\mu\right)^2}{\left(e^a-1\right)^2\left(1+\mu\right)^2}=\frac{e^{(\overline{u_2}-2)a}}{(e^a-1)^2}>1,\end{aligned}$$

and the Claim 2 is proved.

Let $l_a=\min\{A,B\}$, see (2.2.8) for the definition of A and (2.2.11) for the definition of B. Thus, for every $t\in(0,t_0]$ and every $x\in D_{a,t}$, with $a\in(0,\log 2)$, there is a chain $\mathfrak{b}_x$ such that $\left(\Gamma_{a,t}^{\mathfrak{b}_x}\right)'(x)>l_a$, which is the desired conclusion. □

The key step towards proving Theorem 2.1 is presented in the following result.

Proposition 2.7. *For every interval $I=[i_1,i_2]\subset(0,1/2]$, there are $x\in[i_1,i_2)$ and a finite sequence $(\vartheta_0,\vartheta_1,\cdots,\vartheta_r)$, $\vartheta_i\in\{0,1\}$ such that $g_{[\vartheta_0\vartheta_1\cdots\vartheta_r],t}(x)=0$. Thus, for every sequence $(\eta_{-i})_{i\in\mathbb{N}}$ with $\eta_{-i}\in\{0,1\}$,*

$$X=(\cdots\eta_{-1}.\vartheta_0\vartheta_1\cdots\vartheta_r0^{\mathbb{N}},x)\in W^s(Q,G_{a,t}).$$

Proof. Given $I=[i_1,i_2]\subset(0,1/2]$ choose $m\in\mathbb{N}$ such that $g_0^m(I)\subseteq[1/2-t,1/2]$. If $g_0^m(i_1)=1/2-t$ we are done. Otherwise, let $n(I)$ be such that

$$\left(g_0^{n(I)}\circ g_{1,t}\circ g_0^m(I)\right)\cap\left(g_a^k(d_{a,t}),g_a^{k+1}(d_{a,t})\right]\neq\emptyset.$$

Since g_0 and $g_{1,t}$ preserve the orientation, we have either there is $x\in[i_1,i_2)$ such that $g_{[0^m10^{n(I)}],t}(x)=g_a^k(d_{a,t})$, or, by the Corollary 2.6 , there is a chain $\mathfrak{b}$ such that

$$g_a^k(d_{a,t})\in\Gamma_{a,t}^{\mathfrak{b}}\left(g_{[0^m10^{n(I)}],t}(I)\right),$$

which implies that there is $x \in [i_1, i_2)$ such that $g_{[0^m 1 0^{n(I)} \theta(\mathfrak{b}) 0^{n-k} 1],t}(x) = 0$. □

Now we are in position to prove Theorem 2.1. Part of the proof was inspired in [D95b].

Proof of Theorem 2.1. First we prove that $H_{\mathcal{V}}(P, G_{a,t}) \subseteq H_{\mathcal{V}}(Q, Ga, t)$. The main idea of the proof is to prove that any point $X \in W^s(P, G_{a,t}) \pitchfork W^u(P, G_{a,t})$ whose orbit is contained in $\mathcal{V}$ is accumulated by homoclinic points of Q.

Let $X = (\xi, x) \in \Big(W^s(P, G_{a,t}) \pitchfork W^u(P, G_{a,t})\Big) \cap \Lambda_t$. Then we can assume that

$$X = \Big(0^{-\mathbb{N}} 1 0^{-(k+1)}.\xi_0 \cdots \xi_{i_0} 0^{\mathbb{N}}, g_a^{k+1}(t)\Big), \quad \text{with } i_0 \in \mathbb{N},$$

because, if it is not, then some iterate of X is of this form.

Denote by $I_m := \big[g_a^{k+1}(t) - 1/m, g_a^{k+1}(t)\big]$, with $m \in \mathbb{N}$ large enough, such that $g_{[\xi_0 \cdots \xi_m],t}(I_m)$ is also a closed interval. By Proposition 2.7, there are $x_m \in g_{[\xi_0 \cdots \xi_m]}(I_m)$, $x_m \neq g_{[\xi_0 \cdots \xi_m]}(g_a^{k+1}(t))$, and a finite sequence $(\vartheta_0, \cdots, \vartheta_l)$, with $\vartheta_i \in \{0, 1\}$, such that $g_{[\vartheta_0 \cdots \vartheta_l],t}(x_m) = 0$, consequently

$$X_m := \Big(0^{-\mathbb{N}} 1 0^{-(k+1)} \xi_0 \cdots \xi_m.\vartheta_0 \cdots \vartheta_l 0^{\mathbb{N}}, x_m\Big) \in W^s(Q, G_{a,t})$$

and thus

$$\begin{aligned} G_{a,t}^{-m}(X_m) &= \Big(0^{-\mathbb{N}} 1 0^{-(k+1)}.\xi_0 \cdots \xi_m \vartheta_0 \cdots \vartheta_l 0^{\mathbb{N}}, g_{[\xi_0 \cdots \xi_m .],t}(x_m)\Big) \\ &:= \Big(\overline{\zeta_m}, y_m\Big) \in W^s(Q, G_{a,t}), \end{aligned}$$

Consider a open interval J_m containing y_m. Since

$$g_{[1 0^{-(k+1)}.]}(z) = g_{1,t}^{-1} \circ g_a^{-(k+1)}(z) \in (0, 1/2) \text{ for all } z \in J_m,$$

it follows that $\overline{\zeta_m} \times J_m$ is contained in $W^u(Q, G_{a,t})$, then $G_{a,t}^{-m}(X_m)$ is a transverse homoclinic point of Q. Thus $X_m \in \Lambda_{a,t} \cap H_{\mathcal{V}}(Q, G_{a,t})$ and, by construction, $x_m \to x$ and $\overline{\zeta_m} \to \xi$ as $m \to +\infty$, proving the inclusion $H_{\mathcal{V}}(P, G_{a,t}) \subseteq H_{\mathcal{V}}(Q, G_{a,t})$.

It remains to prove that

$$\Lambda_{a,t} = H_{\mathcal{V}}(Q, G_{a,t}), \quad \text{for all } t \notin T.$$

Before that, it is important to note that there are two types of points in $\Lambda_{a,t}$:

1. those points whose orbit does not intersect

$$V_0(k,\epsilon) = [0^{-k}.0^{k}] \times [-\epsilon,\epsilon],$$

   that is points in the set $\{0^{\mathbb{Z}}\} \times (0,1/2) \subset W^s(P,G_{a,t}) \cap W^u(Q,G_{a,t})$ and

2. those points that have some iterate in $V_0(k,\epsilon)$.

We claim that every point $X = (0^{\mathbb{Z}}, x) \in \{0^{\mathbb{Z}}\} \times (0,1/2)$ belongs to $H_{\mathcal{V}}(Q,G_{a,t})$. In fact, for $m \in \mathbb{N}$ large, take $\{0^{-\mathbb{N}}.0^m \cdots\} \times [x-1/m, x+1/m] \subset W^u(Q,G_{a,t})$. By Proposition 2.7 there are $x_m \in [x-1/m, x+1/m]$ and $(\vartheta_0,\cdots,\vartheta_m)$ such that

$$X_m := \left(0^{-\mathbb{N}}.0^m\vartheta_0\cdots\vartheta_m 0^{\mathbb{N}}, x_m\right) \in W^s(Q,G_{a,t})$$

thus $X_m \in H_{\mathcal{V}}(Q,G_{a,t})$. Since $X_m \to X$ as $m \to +\infty$, we have $X \in H_{\mathcal{V}}(Q,G_{a,t})$.

For points $X = (\xi, x) \in \Lambda_t$ of the second type, we can assume that

$$X \in \Delta_{a,t} := [0^{-k}.0^{k}] \times D_{a,t},$$

for if not, we replace X by some iterate of it. From Lemma 1.1, to prove that $X \in H(Q,G_{a,t})$, we consider the following four cases.

case 1. $X \in W^u(P,G_{a,t}) \cup W^u(Q,G_{a,t})$ and has infinitely many forward returns to $\Delta_{a,t}$.

Let $X = (\xi,x) \in W^u(P,G_{a,t}) \cup W^u(Q,G_{a,t})$ with $\xi = (\xi_k)_{k\in\mathbb{Z}} \in \Sigma_2$. For m large, if we consider the sequence $(\xi_0,\ldots,\xi_m)$, by assumptions $X \notin W^s(Q,G_{a,t})$ and $X \in \Lambda_{a,t}$, there is a open interval $I_m = (x-1/m, x+1/m)$ such that $g_{[\xi_0\cdots\xi_m]}(I_m)$ is also an open interval contained on $[0,1/2]$.

If $X \in W^u(P,G_{a,t})$, then the point

$$X_m := \left(\cdots\xi_{-2}\xi_{-1}.\xi_0\xi_1\cdots\xi_m 0^{\mathbb{N}}, x\right) \in W^s(P,G_{a,t}) \pitchfork W^u(P,G_{a,t}),$$

thus $X_m \in H_{\mathcal{V}}(P,G_{a,t}) \subseteq H_{\mathcal{V}}(Q,G_{a,t})$ and $X_m \to X$, as $m \to +\infty$, and, since $H_{\mathcal{V}}(Q,G_{a,t})$ is closed, we get $X \in H_{\mathcal{V}}(Q,G_{a,t})$.

Otherwise, if $X \in W^u(Q, G_{a,t})$, then, replacing X by a backward orbit, we can assume that $X = (0^{-\mathbb{N}}.\xi_0\xi_1\cdots, x)$. By Proposition 2.7 there is $z_m \in g_{[\xi_0\cdots\xi_m]}(I_m)$ and a finite sequence $(\vartheta_0, \ldots, \vartheta_r)$ such that

$$X_m = \big(0^{-\mathbb{N}}.\xi_0\cdots\xi_m\vartheta_0\cdots\vartheta_r 0^{\mathbb{N}}, g_{[\xi_0\cdots\xi_m.]}(z_m)\big) \in H_{\mathcal{V}}(Q, G_{a,t}).$$

and $X_m \to X$, as $m \to \infty$.

case 2. X has infinitely many forward and backward returns $\varrho_i(X)$.

We claim that $X = (\xi, x)$ is accumulated by points $X_m \in W^u(Q, G_{a,t})$. In fact, since $X \notin W^u(P, G_a, t)$ and $X \in \Lambda_{a,t}$ it is enough to take, for m large,

$$X_m = \big(0^{-\mathbb{N}}\xi_{-m}\cdots\xi_{-1}.\xi_0\xi_1\cdots, x\big) \in W^u(Q, G_{a,t}).$$

In the same manner as in case 1 we can conclude that $X_m \in H_{\mathcal{V}}(Q, G_{a,t})$, $X_m \to X$ as $m \to \infty$, and thus $X \in H_{\mathcal{V}}(Q, G_{a,t})$.

case 3. $X \in W^s(P, G_{a,t}) \cup W^s(Q, G_{a,t})$ and has infinitely many backward returns.

Let $X = (\xi, x) \in W^s(P, G_{a,t}) \cap W^s(Q, G_{a,t})$. By a similar way as in case 1 show that $X \in H(Q, G_{a,t})$.

case 4. X has finitely many forward and backward returns to $\Delta_{a,t}$.

Consider $t \notin T$, that is $W^s(Q, G_{a,t}) \cap W^u(P, G_{a,t}) = \emptyset$. Then X must satisfy one of the following three possibilities:

i. $X \in W^s(Q, G_{a,t}) \cap W^u(Q, G_{a,t})$,

ii. $X \in W^s(P, G_{a,t}) \cap W^u(P, G_{a,t})$, and

iii. $X \in W^s(P, G_{a,t}) \cap W^u(Q, G_{a,t})$.

The intersection is transverse in all possibilities. Therefore, in the first case, $X \in H_{\mathcal{V}}(Q, G_{a,t})$, and in the second case, $X \in H_{\mathcal{V}}(P, G_{a,t}) \subseteq H_{\mathcal{V}}(Q, G_{a,t})$.

In the last case, we have

$$X = (0^{-\mathbb{N}}\xi_{-r_0}\cdots\xi_{-1}.\xi_0\cdots\xi_{r_1}0^{\mathbb{N}}, x),$$

with $\xi_i \in \{0,1\}$, $-r_0 \le i \le r_1$. The same conclusion can be drawn by the same method as in $\{0^{\mathbb{Z}}\} \times (0, 1/2) \subset H_{\mathcal{V}}(Q, G_{a,t})$.

Finally, we observe that, by definition, we have $H_{\mathcal{V}}(Q, G_{a,t}) \subseteq \Lambda_{a,t}$. The proof of Theorem 2.1 is now complete. □

Remark 2.8. *From the next proposition, we conclude that*

$$H_{\mathcal{V}}(P, G_{a,t}) \subseteq H_{\mathcal{V}}(Q, G_{a,t}) \text{ if and only if } H_{\mathcal{V}}(Q, G_{a,t}) \subseteq H_{\mathcal{V}}(P, G_{a,t}).$$

Thus, for $0 < a < \log 2$ *and* t *sufficiently small, we have* $H_{\mathcal{V}}(P, G_{a,t}) = H_{\mathcal{V}}(Q, G_{a,t})$.

Proposition 2.9. *For every* $a > 0$ *and* $x \in]0, 1/2[$ *the following properties hold:*
P1) $(g_a^n)'(x) = (g_a^{-n})'(1/2 - x)$;
P2) $g_a^{-m}(x) + g_a^m(1/2 - x) = 1/2$ *for all* $m \in \mathbb{N}$; *and*
P3) $\tilde{d}_{a,t} = 1 - d_{a,t}$, *where* $\tilde{d}_{a,t}$ *is such that* $g_a^{-n}(\tilde{d}_{a,t}) = t$.

Proof. Applying equations (2.1.1), (2.1.2) and (2.2.6) we get

$$\begin{aligned}
(g_a^{-n})'\left(\tfrac{1}{2} - x\right) &= \frac{(1-2t_n)^2}{2t_n^2} \frac{\left(g_a^{-n}\left(\frac{1}{2} - x\right)\right)^2}{\left(\frac{1}{2} - x\right)^2} \\
&= \frac{(1-2t_n)^2}{(2t_n)^2} \frac{(2t_n)^4}{\left(2\left(\frac{1}{2} - x\right)(2t_n)^2 + \left(1 - 2\left(\frac{1}{2} - x\right)\right)(1-2t_n)^2\right)^2} \\
&= \frac{(2t_n)^2(1-2t_n)^2}{\left((1-2x)(2t_n)^2 + 2x(1-2t_n)^2\right)^2} = (g_a^n)'(x),
\end{aligned}$$

then property **P**1) holds.

For $m \in \mathbb{N}$, from the definition of g_a we have

$$g_a^{-m}(x) + g_a^m\left(\frac{1}{2} - x\right) = \frac{x}{2x + (1-2x)e^{ma}} + \frac{\left(\frac{1}{2} - x\right)e^{ma}}{(1-2x)\,e^{ma} + 2x} = \frac{1}{2}.$$

and **P**2) holds.

It remains to prove **P**3). Since $\tilde{d}_{a,t} = g_a^n(t)$ and $d_{a,t} = g_a^{-n}(1/2 - t)$, from **P**2), one has

$$\tilde{d}_{a,t} = g_a^n(t) = \frac{1}{2} - g_a^{-n}\left(\frac{1}{2} - t\right) = \frac{1}{2} - d_{a,t},$$

ending the proof of the proposition. □

The conclusion of Proposition 2.5 remains valid for the skew-products G_t, introduced in Section 1.3, verifying the following condition:

Expansion Condition (EC): *There are $l_t > 1$ and $r \in \mathbb{N}$ such that, for every $x \in (d_t, g_{0,t}(d_t)]$, there is a finite chain $\mathfrak{b} = \mathfrak{b}(y)$ verifying $|\theta(\mathfrak{b})| \le r$ and $\left(\Gamma_t^{\mathfrak{b}}\right)'(x) > l_t$.*

Corollary 2.10. *For each $t > 0$ small enough and under the condition* **(EC)** *we have*

$$H_{\mathcal{V}}(P_t, G_t) \subseteq H_{\mathcal{V}}(Q_t, G_t) \text{ and } \Lambda_t \setminus (W^s(Q_t, G_t) \cap W^u(P_t, G_t)) = H_{\mathcal{V}}(Q_t, G_t).$$

2.3 Persistence of non-hyperbolicity

In this section, we fix $\log 2 < a < \log 4$ and prove Theorem 2.3.

We begin with the observation that $D_{a,t}^{(0,n-2k)} = \emptyset$ for all $a > 0$. Indeed, from the conditions $g_{1,t}(1/2) = t$ and $g_a^k(t) < g_a^k(d_{a,t})$, and the definition of $D_{a,t}$, we get

$$g_a^k \circ g_{1,t} \circ g_a^{n-k}(x) < g_a^k(d_{a,t}), \quad \text{for all } x \in D_{a,t}.$$

Moreover in the first section we proved that for $0 < a < \log 2$, there is a small $t_0 > 0$ such that $D_{a,t}^{(1,n-2k)} = \emptyset$ for all $t \in (0, t_0]$.

Proposition 2.11. *Let $a > \log 2$. Then there is $n_0 = n_0(a) \in \mathbb{N}$ such that, for every $n \ge n_0$, $D_{a,t_n}^{(1,n-2k)} \ne \emptyset$.*

Proof. Let $a > \log 2$. To show that $D_{a,t_n}^{(1,n-2k)} \ne \emptyset$ it is enough to see that

$$\Gamma_{a,t_n}^{(1,n-2k)}(g_a^{k+1}(t_n)) > g_a^k(t_n).$$

By definition of t_n and $\Gamma_{a,t_n}^{(1,n-2k)}$,

$$\Gamma_{a,t_n}^{(1,n-2k)}(g_a^{k+1}(t_n)) = g_a^{k+1} \circ g_{1,t_n} \circ g_a^{n-k}(g_a^{k+1}(t_n)) = g_a^{k+1} \circ g_{1,t_n} \circ g_a\left(\frac{1}{2} - t_n\right),$$

so, since g_a and g_{1,t_n} preserve the orientation, the condition

$$\Gamma_{a,t_n}^{(1,n-2k)}(g_a^{k+1}(t_n)) > g_a^k(t_n)$$

is equivalent to $g_{1,t_n} \circ g_a\,(1/2 - t_n) > g_a^{-1}(t_n)$.

From

$$\begin{aligned} g_{1,t_n} \circ g_a\,(1/2 - t_n) - g_a^{-1}(t_n) &= \frac{(e^a - 1)t_n(1-2t_n)}{2t_n + e^a(1-2t_n)^2} - \frac{t_n}{2t_n + (1-2t_n)e^a} \\ &= \frac{t_n[(e^a-1)(1-2t_n) - 1]}{2t_n + (1-2t_n)e^a} \end{aligned}$$

we obtain

$$\Gamma_{a,t_n}^{(1,n-2k)}(g_a^{k+1}(t_n)) > g_a^k(t_n) \Leftrightarrow (e^a - 1)(1-2t_n) - 1 > 0 \Leftrightarrow a > \log\left(\frac{2-2t_n}{1-2t_n}\right).$$

Choosing $a > \log 2$, there is $n_0 = n_0(a) \in \mathbb{N}$ such that $a > \log\left(\frac{2-2t_{n_0}}{1-2t_{n_0}}\right)$. Therefore, for all $n \geq n_0$, $a > \log\left(\frac{2-2t_n}{1-2t_n}\right)$ and consequently

$$\Gamma_{a,t_n}^{(1,n-2k)}(g_a^{k+1}(t_n)) > g_a^k(t_n), \text{ for all } t_n < t_{n_0},$$

and the proof of the lemma is complete. □

Proposition 2.12. *For* $\log 2 < a < \log 4$ *there is* $n_0 = n_0(a)$ *large such that, for all* $n \geq n_0$*, the map* $\Gamma_{a,t_n}^{(1,n-2k)} = g_a^{k+1} \circ g_{1,t_n} \circ g_a^{n-k}$ *has no fixed points in* $D_{a,t_n}^{(1,n-2k)}$.

Proof. Observe that $\Gamma_{a,t_n}^{(1,n-2k)}$ has no fixed points if and only if

$$\Gamma_{a,t_n}^{(1,n-2k)}(x) < x, \text{ for all } x \in D_{a,t_n}^{(1,n-2k)},$$

which is equivalent to $g_{1,t_n} \circ g_a^{n+1}(g_a^{-(k+1)}(x)) < g_a^{-(k+1)}(x)$. Thus, the main idea is to prove that $g_{1,t_n} \circ g_a^{n+1}(z) < z$ with

$$z \in g_a^{-(k+1)}\left(D_{a,t_n}^{(1,n-2k)}\right) \subset \left(g_a^{-1}(t_n), t_n\right].$$

The procedure is to find $t \in [t_n, t_{n-1})$, close to t_n, such that $g_{1,t} \circ g_a^{n+1}$ has a saddle node in $(g_a^{-1}(t_n), t_n]$. Since

$$g_{1,t_n} \circ g_a^{n+1}(x) < g_{1,t} \circ g_a^{n+1}(x), \ \forall x \in (g_a^{-1}(t_n), t_n]$$

we can conclude that that $g_{1,t_n} \circ g_a^{n+1}$ has no fixed points.

First we compute $z^\star$ such that $(g_{1,t_n} \circ g_a^{n+1})'(z^\star) = (g_a^{n+1})'(z^\star) = 1$. Note that

$$(g_a^{n+1})'(z) = 1 \quad \Leftrightarrow \quad e^{-a}\frac{(2t_n)^2}{(1-2t_n)^2}\frac{(g_a^{n+1})^2(z)}{z^2} = 1$$

$$\Leftrightarrow \quad \left(\frac{g_a^{n+1}(z)}{z}\right)^2 = \frac{e^a(1-2t_n)^2}{(2t_n)^2}.$$

Thus, by equation (2.2.5) we obtain

$$(g_{1,t_n} \circ g_a^{n+1})'(z) = 1 \quad \Rightarrow \quad \frac{(1-2t_n)^2}{2z(1-2t_n)^2 + (1-2z)(2t_n)^2 \cdot e^{-a}} = \frac{e^{\frac{a}{2}} \times (1-2t_n)}{2t_n}$$

$$\Rightarrow \quad z[(1-2t_n)^2 e^a - (2t_n)^2] = t_n(1-2t_n)e^{\frac{a}{2}} - 2t_n^2$$

and solving it in order to z, it follows

$$z^\star = \frac{t_n\Big((1-2t_n)e^{\frac{a}{2}} - 2t_n\Big)}{(1-2t_n)^2 e^a - (2t_n)^2} = \frac{t_n}{(1-2t_n)e^{\frac{a}{2}} + 2t_n}, \tag{2.3.1}$$

observing that

$$\lim_{n\to+\infty} \frac{z^\star - t_n}{t_n} = e^{-\frac{a}{2}} - 1 < 0$$

and

$$\lim_{n\to+\infty} \frac{z^\star - g_a^{-1}(t_n)}{t_n} = e^{-\frac{a}{2}} - e^{-a} > 0,$$

we conclude that, for n large, $z^\star \in [g_a^{-1}(t_n), t_n]$.

Next we find t such that $g_{1,t} \circ g_a^{n+1}(z^\star) = z^\star$. Replacing this value of $z^\star$ in equation (2.2.5), one gets

$$\begin{aligned} g_a^{n+1}(z^\star) &= \frac{z^\star(1-2t_n)^2}{2z^\star(1-2t_n)^2 + (1-2z)(2t_n)^2 \cdot e^{-a}} \\ &= \frac{t_n(1-2t_n)^2}{2t_n(1-2t_n)^2 + (1-2t_n)e^{\frac{a}{2}}(2t_n)^2 e^{-a}} \\ &= \frac{1-2t_n}{2(1-2t_n) + 4t_n e^{-\frac{a}{2}}}. \end{aligned}$$

Thus the equation $g_a^{n+1}(z^\star) - 1 + t = z^\star$ is equivalent to

$$\frac{(1-2t_n)}{2(1-2t_n)+4t_ne^{-\frac{a}{2}}} - 1 + t = \frac{t_n}{(1-2t_n)e^{\frac{a}{2}}+2t_n}$$

$$\Rightarrow \quad \frac{-t_n}{(1-2t_n)e^{\frac{a}{2}}+2t_n} + t = \frac{t_n}{(1-2t_n)e^{\frac{a}{2}}+2t_n}$$

$$\Rightarrow \quad t = \frac{2t_n}{(1-2t_n)e^{\frac{a}{2}}+2t_n},$$

Since $t_{n-1} = e^{\frac{a}{2}}t_n/(2t_ne^{\frac{a}{2}} + (1-2t))$ (see (2.2.3)), by computations we can show that

$$t_n < t < t_{n-1} \Leftrightarrow t_n < \frac{2t_n}{(1-2t_n)e^{\frac{a}{2}}+2t_n} < t_{n-1}$$

is equivalent to

$$\log\left(\frac{t_n+\sqrt{9t_n^2-8t_n+2}}{1-2t_n}\right)^2 < a < \log\left(\frac{2-2t_n}{1-2t_n}\right)^2,$$

and the proof is concluded. □

Note that, from the previous proposition, we can conclude that, for $a > \log 4$, there are $n_0 = n_0(a) > 0$ large and $\mu^\star_{a,t_n} \in (t_{n+1}, t_n)$ such that the map $\Gamma_{a,t}^{(1,n-2k)}$ has two fixed points, for all $n \geq n_0$ and $t \in (\mu^\star_{a,t_n}, t_n)$. We will consider this situation in the last chapter.

Now we introduce some notations. Define

$$H(a,t_n) := \bigcup_{i\geq 2} D_{a,t_n}^{(i,n-2k)} = (g_a^k(t_n), d_{1,t_n}], \tag{2.3.2}$$

where

$$d_{1,t_n} := \left(\Gamma_{a,t_n}^{(1,n-2k)}\right)^{-1}(g_a^k(t_n)).$$

It is obvious that $D_{a,t_n} = H(a,t_n) \cup D_{a,t_n}^{(1,n-2k)}$, and, since (2.3.2) is a disjoint union, we can define the following map on $H(a,t_n)$

$$\begin{array}{rccl} \Upsilon_{a,t_n}: & H(a,t_n) & \to & (g_a^k(t_n), g_a^{k+1}(t_n)], \\ & D_{a,t_n}^{(i,n-2k)} \ni x & \mapsto & \Gamma_{a,t_n}^{(i,n-2k)}(x). \end{array}$$

Since the graph of $\Gamma_{a,t_n}^{(1,n-2k)}$ is below the diagonal, for n large, there is a first $l_0 \in \mathbb{N}$ such that $\left(\Gamma_{a,t_n}^{(1,n-2k)}\right)^{l_0}(g_a^{k+1}(t_n)) \in H_{a,t_n}$. We define the set

$$I_l(a,t_n) := \left(\Gamma_{a,t_n}^{(1,n-2k)}\right)^{-l}(H(a,t_n)) \cap D_{a,t_n},$$

for each $l \in \{1, \ldots, l_0\}$, so

$$D_{a,t_n}^{(1,n-2k)} = \bigcup_{i=0}^{l_0} I_i(a,t_n)$$

is also a disjoint union.

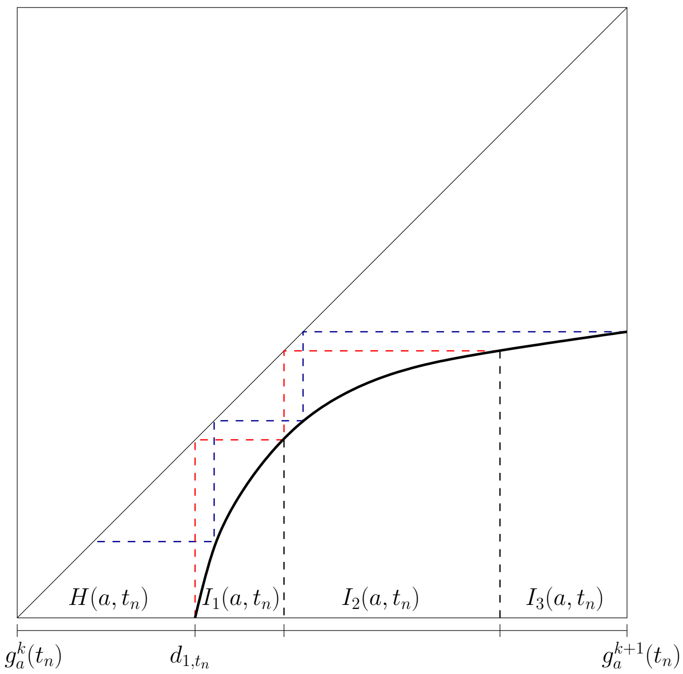

Figure 2.4: The sets $H(a,t_n)$ and $I_l(a,t_n)$ with $l_0 = 3$

We also define the map

$$\begin{array}{rcl} \Psi_{a,t_n}: \; D_{a,t_n}^{(1,n-2k)} & \rightarrow & (g_a^k(t_n), g_a^{k+1}(t_n)] \\ I_l(a,t_n) \ni x & \mapsto & \Upsilon_{a,t_n} \circ \left(\Gamma_{a,t_n}^{(1,n-2k)}\right)^l(x). \end{array}$$

The next lemma is the key step for obtaining non-hyperbolicity for a set of parameters $J(a,t_n)$.

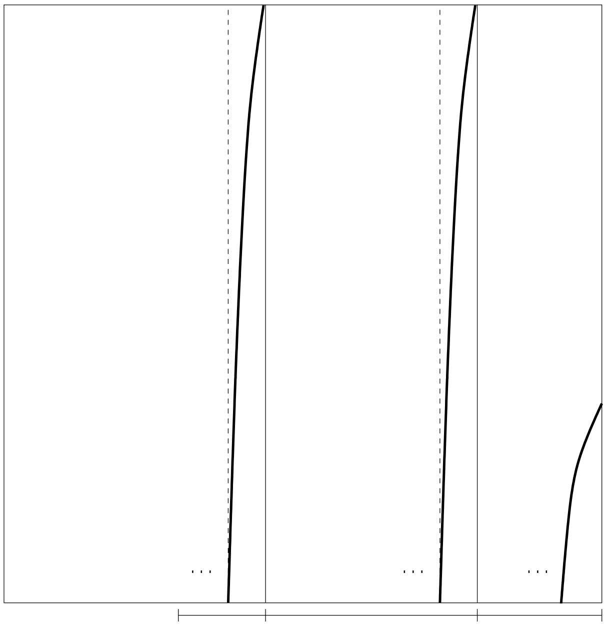

Figure 2.5: The map Ψ_{a,t_n}

Lemma 2.13. *Assuming that* $\left(\Gamma^{(1,n-2k)}_{a,t_n}\right)^{l_0}(g_a^{k+1}(t_n)) = d_{1,t_n}$, *we have that*

$$\left(g_a \circ \left(\Gamma^{(1,n-2k)}_{a,t_n}\right)^{l_0+1}\right)'(g_a^{k+1}(t_n)) = (\Psi_{a,t_n})'(g_a^{k+1}(t_n)) = 1.$$

Moreover, if $\left(\Gamma^{(1,n-2k)}_{a,t_n}\right)^{l_0}(g_a^{k+1}(t_n)) < d_{1,t_n}$, *then* $(\Psi_{a,t_n})'(g_a^{k+1}(t_n)) > 1$ *and, for each* $i = 1, \ldots, l_0$, $(\Psi_{a,t_n})'\left((\Gamma^{(1,n-2k)}_{a,t_n})^{-i}(g_a^k(t_n))\right) > 1$.

As a consequence of the previous lemma, which the proof is presented at the end of this chapter, we have the following result:

Proposition 2.14. *Let* $a \in (\log 2, \log 4)$. *Then there is* $n_0(a) \in \mathbb{N}$ *large enough such that, for every* $n \geq n_0(a)$, *there exists* $l_{a,t_n} > 1$ *satisfying the following condition: for every* $x \in (g_a^{-1}(t_n), t_n]$ *there is a chain* $\mathfrak{b} = \mathfrak{b}(x)$ *such that* $\left(\Gamma^{\mathfrak{b}}_{a,t_n}\right)'(x) > l_{a,t_n}$.

Proof. First we consider $x \in H(a,t_n)$. Thus, by definition, there is $i \geq 2$ such that $x \in D^{(i,n-2k)}_{a,t_n}$. We claim that

$$\left(\Gamma^{(i,n-2k)}_{a,t}\right)'(x) = (\Upsilon_{a,t_n})'(x) > 1.$$

Indeed, since g'_a is a decreasing map on $[0,1/2]$, it is enough to prove that $\Upsilon'(d_{1,t_n}) > 1$. By an easy computation we can show that

$$\lim_{n\to\infty} (\Upsilon_{a,t_n})'(d_{1,t_n}) = (e^a - 1)^2 > 1, \quad \text{for } a > \log 2.$$

Now consider $x \in D^{(1,n-2k)}_{a,t_n}$ and, as above, let l_0 denote the first natural number such that $(\Gamma^{(1,n-2k)}_{a,t_n})^{l_0}(g_a^{k+1}(t_n)) \in H(a,t_n)$. Since g_a and g_{1,t_n} are strictly increasing and g'_a is strictly decreasing, it suffices to show that

$$(\Psi_{a,t_n})'\left((\Gamma^{(1,n-2k)}_{a,t_n})^{-(l_0-1)}(d_{1,t_n})\right) > 1 \text{ and } (\Psi_{a,t_n})'(g_a^{k+1}(t_n)) > 1.$$

There are two cases to be considered,

$$\lim_{n\to\infty} \frac{(\Gamma^{(1,n-2k)}_{a,t_n})^{l_0}(g_a^{k+1}(t_n)) - d_{1,t_n}}{t_n} = 0$$

and

$$\lim_{n\to\infty} \frac{(\Gamma^{(1,n-2k)}_{a,t_n})^{l_0}(g_a^{k+1}(t_n)) - d_{1,t_n}}{t_n} < 0.$$

Case 1. $\displaystyle\lim_{n\to\infty} \frac{(\Gamma^{(1,n-2k)}_{a,t_n})^{l_0}(g_a^{k+1}(t_n)) - d_{1,t_n}}{t_n} = 0$

In this case, there is n_0 large and $a^\star_{n_0} > a$ such that

$$(\Gamma^{(1,n_0-2k)}_{a^\star_{n_0},t_{n_0}})^{l_0}(g^{k+1}_{a^\star_{n_0}}(t_{n_0})) = d_{1,t_{n_0}}.$$

Then

$$\Gamma^{(2,n_0-2k)}_{a^\star_{n_0},t_{n_0}} \circ \left(\Gamma^{(1,n_0-2k)}_{a^\star_{n_0},t_{n_0}}\right)^{l_0}(g^{k+1}_{a^\star_{n_0}}(t_{n_0})) = g_{a^\star_{n_0}} \circ (\Gamma^{(1,n_0-2k)}_{a^\star_{n_0},t_{n_0}})^{l_0}(g^{k+1}_{a^\star_{n_0}}(t_{n_0})) = g^{k+1}_{a^\star_{n_0}}(t_{n_0})$$

and, by Lemma 2.13,

$$(\Psi_{a^\star_{n_0},t_{n_0}})'(g^{k+1}_{a^\star_{n_0}}(t_{n_0})) = 1.$$

In fact $\Gamma^{(2,n_0-2k)}_{\alpha,t_{n_0}} \circ (\Gamma^{(1,n_0-2k)}_{\alpha,t_{n_0}})^{l_0}$ has two fixed points for $\alpha > a^\star_{n_0}$ and it does not have fixed points for $\alpha < a^\star_{n_0}$, thus $\Gamma^{(2,n_0-2k)}_{\alpha,t_{n_0}} \circ (\Gamma^{(1,n_0-2k)}_{\alpha,t_{n_0}})^{l_0}$ has a saddle-node at the parameter $\alpha = a^\star_{n_0}$.

For $n \geq n_0$ we have $(\Gamma_{a,t_n}^{(1,n-2k)})^{l_0}(g_a^{k+1}(t_n)) < d_{1,t_n}$ and, again by Lemma 2.13,

$$(\Psi_{a,t_n})'(x) \geq \min\left\{ (\Psi_{a,t_n})'(g_a^{k+1}(t_n)),\ (\Psi_{a,t_n})'\left((\Gamma_{a,t_n}^{(1,n-2k)})^{-(l_0-1)}(d_{1,t_n})\right) \right\} > 1,$$

for all $x \in D_{a,t_n}^{(1,n-2k)}$.

Case 2. $\lim_{n\to\infty} \frac{(\Gamma_{a,t_n}^{(1,n-2k)})^{l_0}(g_a^{k+1}(t_{n_0})) - d_{1,t_n}}{t_n} < 0$

In this case, from the proof of Lemma 2.13, presented below, we have

$$\lim_{n\to\infty} (\Psi_{a,t_n})'(g_a^{k+1}(t_n)) > 1 \text{ and } \lim_{n\to\infty} (\Psi_{a,t_n})'\left((\Gamma_{a,t_n}^{(1,n-2k)})^{-(l_0-1)}(d_{1,t_n})\right) > 1,$$

thus there is $t_0 = t_0(a) > 0$ such that, for all $t_n \in (0, t_0]$, $(\Psi_{a,t_n})'(x) \geq l_a$, with l_a satisfying

$$\min\left\{ (\Upsilon_{a,t_n})'(d_{1,t_n}), (\Psi_{a,t_n})'(g_a^{k+1}(t_n)), (\Psi_{a,t_n})'\left((\Gamma_{a,t_n}^{(1,n-2k)})^{-(l_0-1)}(d_{1,t_n})\right) \right\} > l_a > 1.$$

Consequently in the second case l_a is independent of t_n and the proof is completed. □

Since the function $a \mapsto \Gamma_{a,t_n}^{(1,n-2k)}$ is increasing and, from the proof of Proposition 2.12, we know that for $a > \log 4$ there is n_0 such that $\Gamma_{a,t_n}^{(1,n-2k)}$ has two fixed points, for all $n \geq n_0$, we conclude that the sequence $(a_l)_{l\in\mathbb{N}}$ satisfy $a_l \to \log 4$, as $l \to +\infty$, where, for each $l \in \mathbb{N}$, a_l denote the unique solution on $(\log 2, \log 4)$ of the equation

$$\lim_{n\to\infty} \frac{\left(\Gamma_{a,t_n}^{(1,n-2k)}\right)^{l} (g_a^{k+1}(t_n)) - d_{1,t_n}}{t_n} = 0.$$

Set $a_0 := \log 2$. For $a \in (\log 2, \log 4)$, there is $l_0 \in \mathbb{N}$ such that $a_{l_0-1} \leq a \leq a_{l_0}$. By the proof of Proposition 2.14, there is $t_0 = t_0(a)$ small (in addiction $t_0(a) \to 0$ as $a \to a_{l-1}$) such that for all $t_n \in (0, t_0)$ the following conditions hold:

(P1) $\left(\Gamma_{a,t_n}^{(1,n-2k)}\right)^{l_0} (g_a^{k+1}(t_n)) \in \left(g_a^k(t_n), d_{1,t_n}\right)$;

(P2) $(\Psi_{a,t_n})'(g_a^{k+1}(t_n)) > 1$;

(P3) $(\Psi_{a,t_n})'\left((\Gamma_{a,t_n}^{(1,n-2k)})^{-(l_0-1)}(d_{1,t_n})\right) > 1$; and

(P4) $(\Upsilon_{a,t_n})'(d_{1,t_n}) > 1$.

Since (P1)-(P4) are open conditions, for each $n \geq n_0$, by continuity we can conclude that (P1)-(P4) also hold replacing t_n by t for t close to t_n. This means that we can consider, for each n large, intervals

$$J(a,t_n) = (t_n - \alpha_{a,t_n}, t_n + \alpha_{a,t_n}), \tag{2.3.3}$$

where

(P1') $\left(\Gamma_{a,t}^{(1,n-2k)}\right)^{l_0}(g_a^{k+1}(d_{a,t})) \in \left(g_a^k(d_{a,t}), d_{1,t}\right)$;

(P2') $(\Psi_{a,t})'(g_a^{k+1}(d_{a,t})) > 1$;

(P3') $(\Psi_{a,t})'\left((\Gamma_{a,t}^{(1,n-2k)})^{-(l_0-1)}(d_{1,t})\right) > 1$; and

(P4') $(\Upsilon_{a,t})'(d_{1,t}) > 1$

holds for all $t \in J(a,t_n)$.

Now, the Proposition 2.14 implies that, for every $t \in J(a,t_n)$, there is $l_{a,t} > 1$, such that for each $x \in D_{a,t} = (g_a^k(d_{a,t}), g_a^{k+1}(d_{a,t})]$, we can find a finite chain $\mathfrak{b} = \mathfrak{b}(x)$ verifying

$$(\Gamma_{a,t}^{\mathfrak{b}})'(x) > l_{a,t} > 1,$$

and the proof of Theorem 2.3 now follows arguing exactly as in the Theorem 2.1.

Let us now prove the main lemma.

Proof of Lemma 2.13. We begin by take $l_0 \in \mathbb{N}$ such that $g^{k+1}(t_n) \in I_{l_0}(a,t_n)$. The proof is divided into several steps. First we solve the equation $h'_{a,t_n}(x)h'_{a,t_n}(z) = 1$, where

$$\begin{array}{rrcl} h_{a,t_n}: & [g_a^{-1}(t_n), t_n] & \to & [0,t_n) \\ & x & \mapsto & h_{a,t_n}(y) = g_{1,t_n} \circ g_a^{n+1}(x). \end{array}$$

Then we need to compute the inverse of h_{a,t_n} to prove, by induction on m, that

$$h'_{a,t_n}(h^m_{a,t_n}(t_n)) = \frac{1}{h'_{a,t_n}(h^{-m}_{a,t_n}(g_a^{-1}(t_n)))}, \ \forall m \in \mathbb{N}.$$

Afterwards we show that

$$\Gamma_{a,t_n}^{(2,n-2k)} \circ \left(\Gamma_{a,t_n}^{(1,n-2k)}\right)^{l_0}(g_a^{k+1}(t_n)) = g_a^{k+1}(t_n) \text{ implies } (\Psi_{a,t_n})'(g_a^{k+1}(t_n)) = 1,$$

and finally we prove the second part of the lemma.

Claim 1. If $h'_{a,t_n}(x)h'_{a,t_n}(z) = 1$, then

$$z = \frac{t_n^2(1-2x)}{e^a(1-2t_n)^2x + 2(1-2x)t_n^2}.$$

Note that $h'_{a,t_n} = (g_a^{n+1})'$. By (2.1.2), one has

$$(g_a^{n+1})'(x) \times (g_a^{n+1})'(z) = 1 \;\;\Rightarrow\;\; \left(\frac{g_a^{n+1}(x)}{x}\right)^2 \times \left(\frac{g_a^{n+1}(x)}{x}\right)^2 = e^{2a}\frac{(1-2t_n)^4}{(2t_n)^4}$$

$$\Rightarrow\;\; \frac{g_a^{n+1}(z)}{z} = e^a\frac{(1-2t_n)^2}{(2t_n)^2} \times \frac{x}{g_a^{n+1}(x)}.$$

From (2.1.1) and (2.2.5) we have

$$h'_{a,t_n}(x) \times h'_{a,t_n}(z) = 1$$

$$\Leftrightarrow\;\; \frac{(1-2t_n)^2 \cdot e^a}{2z(1-2t_n)^2 \cdot e^a + (1-2z)(2t_n)^2} = \frac{x(1-2t_n)^2 \cdot e^a + (1-2x)2t_n^2}{2t_n^2}$$

$$\Leftrightarrow\;\; 2z(1-2t_n)^2 \cdot e^a + (1-2z) \times (2t_n)^2 = \frac{2t_n^2(1-2t_n)^2 \cdot e^a}{e^a(1-2t_n)^2x + (1-2x)2t_n^2}$$

$$\Leftrightarrow\;\; 2z\left[(1-2t_n)^2 \cdot e^a - (2t_n)^2\right] + (2t_n)^2 = \frac{2t_n^2(1-2t_n)^2e^a}{e^a(1-2t_n)^2x + (1-2x)2t_n^2}$$

$$\Leftrightarrow\;\; 2z\left[(1-2t_n)^2 \cdot e^a - (2t_n)^2\right] = \frac{2t_n^2(1-2t_n)^2(1-2x) \cdot e^a - 2^3t_n^4(1-2x)}{e^a(1-2t_n)^2x + 2(1-2x)t_n^2}$$

$$\Leftrightarrow\;\; 2z\left[(1-2t_n)^2 \cdot e^a - (2t_n)^2\right] = \frac{2t_n^2(1-2x)\left[(1-2t_n)^2 \cdot e^a - (2t_n)^2\right]}{e^a(1-2t_n)^2x + (1-2x)2t_n^2}$$

Therefore, since $(1-2t_n)^2 - e^{-a}(2t_n)^2 \neq 0$,

$$z = \frac{t_n^2(1-2x)}{e^a(1-2t_n)^2x + 2(1-2x)t_n^2},$$

and the claim is proved.

Now we compute the inverse of h_{a,t_n}, which is

$$h_{a,t_n}^{-1}(x) = \frac{t_n^2(2x+1-2t_n)}{e^a(1-2t_n)^2(t_n-x)+2t_n^2(1-2t_n+2x)}. \tag{2.3.4}$$

First we note that

$$\begin{aligned} h_{a,t_n}(x) &= g_a^{n+1}(x) - \frac{1}{2} + t_n \\ &= \frac{e^a(1-2t_n)^2 x}{2x(1-2t_n)^2 e^a + (1-2x)(2t_n)^2} - \frac{1}{2} + t_n, \end{aligned}$$

and

$$\begin{aligned} h_{a,t_n}(x) = z &\Leftrightarrow \frac{e^a(1-2t_n)^2 x}{x(1-2t_n)^2 e^a + 2(1-2x)t_n^2} = 2z+1-2t_n \\ &\Leftrightarrow e^a(1-2t_n)^2 x = (2z+1-2t)\left(x\left[(1-2t_n)^2 e^a - 4t_n^2\right] + 2t_n^2\right) \\ &\Leftrightarrow x = \frac{t_n^2(2z+1-2t_n)}{e^a(1-2t_n)^2(t_n-z)+2t_n^2(1-2t_n+2z)}, \end{aligned}$$

as required.

Claim 2. We have

$$h'_{a,t_n}(h_{a,t_n}^m(t_n)) = \frac{1}{h'_{a,t_n}(h_{a,t_n}^{-m}(g_a^{-1}(t_n)))}, \ \forall m \in \mathbb{N} \tag{2.3.5}$$

We use induction on m to prove the claim.
For $m=1$ we need to prove that

$$h'_{a,t_n}(h_{a,t_n}(t_n)) \times h'_{a,t_n}(h_{a,t_n}^{-1}(g_a^{-1}(t_n))) = 1.$$

By the Claim 1 it suffices to show that

$$h_{a,t_n}^{-1}(g_a^{-1}(t_n)) = \frac{t_n^2(1-2h_{a,t}(t_n))}{e^a(1-2t_n)^2 h_{a,t}(t_n) + 2(1-2h_{a,t}(t_n))t_n^2}. \tag{2.3.6}$$

On the one hand, since $g_a^n(t_n) = 1/2 - t_n$ we have

$$\begin{aligned} h_{a,t_n}(t_n) &= g_a\left(\frac{1}{2} - t_n\right) - \frac{1}{2} + t_n \\ &= \frac{t_n(1-2t_n)(e^a-1)}{(1-2t_n)e^a + 2t_n} \end{aligned}$$

and

$$\begin{aligned} 1 - 2h_{a,t}(t_n) &= 1 - 2g_a\left(\frac{1}{2} - t_n\right) + 1 - 2t_n \\ &= 2(1-t_n) - \frac{(1-2t_n)e^a}{(1-2t_n)e^a + 2t_n} \\ &= \frac{(1-2t_n)^2e^a + 4(1-t_n)t_n}{(1-2t_n)e^a + 2t_n}. \end{aligned}$$

On the other hand, using (2.3.4), one gets

$$\begin{aligned} h_{a,t_n}^{-1}\left(g_a^{-1}(t_n)\right) &= h_{a,t_n}^{-1}\left(\frac{t_n}{2t_n + (1-2t_n)e^a}\right) \\ &= \frac{t_n^2((1-2t_n)^2e^a + 4(1-t_n)t_n)}{e^a(1-2t_n)^2t_n(1-2t_n)(e^a-1) + 2((1-2t_n)^2e^a + 4(1-t_n)t_n)t_n^2} \end{aligned}$$

and (2.3.6) follows.

Now we assume that (2.3.5) holds for m, i.e.,

$$h'_{a,t}(h_{a,t_n}^m(t_n)) = \frac{1}{h'_{a,t_n}(h_{a,t_n}^{-m}(g_a^{-1}(t_n)))},$$

and we need to prove (2.3.5) for $m+1$, that is,

$$h'_{a,t}(h_{a,t}^{m+1}(t_n)) = \frac{1}{h'_{a,t}(h_{a,t}^{-m-1}(g_a^{-1}(t_n)))} \tag{2.3.7}$$

Considering $x = h_{a,t_n}^m(t_n)$ in Claim 1, we obtain

$$h_{a,t_n}^{-m}(g_a^{-1}(t_n)) = \frac{t_n^2(1-2x)}{e^a(1-2t_n)^2x + 2(1-2x)t_n^2}$$

and, again by Claim 1, the proof is completed if we show

$$h_{a,t}^{-1}\left(\frac{t_n^2(1-2x)}{e^a(1-2t_n)^2x+2(1-2x)t_n^2}\right)$$

$$= \frac{t_n^2(1-2h_{a,t}(x))}{e^a(1-t_n)^2h_{a,t_n}(x)+2t_n^2(1-2h_{a,t_n}(x))}. \tag{2.3.8}$$

From the definition of h_{a,t_n}, we have

$$h_{a,t_n}(x) = \frac{t_n(1-2t_n)[e^a(1-2t_n)x-(1-2x)t_n]}{e^a(1-2t_n)^2x+2(1-2t_n)t_n^2}$$

and

$$1-2h_{a,t_n}(x) = \frac{(1-2t)^3xe^a+(2t)^2(1-t)(1-2x)}{e^a(1-2t_n)^2x+2(1-2t_n)t_n^2}.$$

By computing the left hand side of (2.3.8), we obtain the right hand side with $h_{a,t_n}(x)$ and $1-2h_{a,t_n}(x)$ above calculated and the Claim 2 follows.

We also have

$$(g_a^{n-k})'(g_a^{k+1}(t_n)) = \frac{(2e^{(k+1)a}t+(1-2t))^2}{e^{ak}(2t+e^a(1-2t_n))^2} = \frac{1}{(g_a^{k+2})'\left(g_a^{-1}(t_n)\right)}. \tag{2.3.9}$$

Assuming

$$\left(\Gamma_{a,t_n}^{(1,n-2k)}\right)^{l_0}(g_a^{k+1}(t_n)) = d_{1,t_n}$$

$$\Leftrightarrow \quad \Gamma_{a,t_n}^{(2,n-2k)}\circ\left(\Gamma_{a,t_n}^{(1,n-2k)}\right)^{l_0}(g_a^{k+1}(t_n)) = g_a^{k+1}(t_n), \tag{2.3.10}$$

we claim that $(\Psi_{a,t_n})'(g_a^{k+1}(t_n)) = 1$.

Since

$$\Gamma_{a,t_n}^{(1,n-2k)}\circ\Gamma_{a,t_n}^{(1,n-2k)} = g_a^{k+1}\circ g_{1,t}\circ g_a^{n-k}\circ g_a^{k+1}\circ g_{1,t}\circ g_a^{n-k}$$

$$= g_a^{k+1}\circ h_{a,t_n}\circ g_{1,t}\circ g_a^{n-k},$$

one has

$$\Gamma_{a,t_n}^{(2,n-2k)}\circ\left(\Gamma_{a,t_n}^{(1,n-2k)}\right)^{l_0}(g_a^{k+1}(t_n)) = g_a^{k+2}\circ h_{a,t_n}^{l_0}\circ g_{1,t}\circ g_a^{n-k}(g_a^{k+1}(t_n)),$$

thus

$$\Gamma^{(2,n-2k)}_{a,t_n} \circ \left(\Gamma^{(1,n-2k)}_{a,t_n}\right)^{l_0} (g_a^{k+1}(t_n)) = g_a^{k+2} \circ h^{l_0+1}_{a,t_n}(t_n). \tag{2.3.11}$$

From (2.3.10) we get

$$g_a^{k+2} \circ h^{l_0+1}_{a,t_n}(t_n) = g_a^{k+1}(t_n) \quad \Rightarrow \quad h^{l_0+1}_{a,t_n}(t_n) = g_a^{-1}(t_n) \tag{2.3.12}$$

Using (2.3.9), (2.3.10) and (2.3.11), we obtain

$$\begin{aligned}
\left(\Psi_{a,t_n}\right)' (g_a^{k+1}(t_n)) &= \left(g_a^{k+2} \circ h^{l_0}_{a,t_n} \circ g_{1,t} \circ g_a^{n-k}\right)' (g_a^{k+1}(t_n)) \\
&= (g_a^{k+2})'(g_a^{-1}(t_n))(h^{l_0}_{a,t_n})'(h_{a,t_n}(t_n)) \cdot (g_a^{n-k})'(g_a^{k+1}(t_n)) \\
&= (h^{l_0}_{a,t_n})'(h_{a,t_n}(t_n)) \\
&= (h_{a,t_n})'(h_{a,t_n}(t_n))(h_{a,t_n})'(h^2_{a,t_n}(t_n)) \cdots (h_{a,t_n})'(h^{l_0}_{a,t_n}(t_n))
\end{aligned}$$

and thus, combining the Claim 2 with (2.3.12) yields $\left(\Psi_{a,t_n}\right)' (g_a^{k+1}(t_n)) = 1$.

In order to prove the second part of the lemma, we assume that

$$\left(\Gamma^{(1,n-2k)}_{a,t_n}\right)^{l_0} (g_a^{k+1}(t_n)) < d_{1,t_n}.$$

In this situation we have

$$\Psi_{a,t_n}(g_a^k(t_n)) = \Gamma^{(i,n-2k)}_{a,t_n} \circ \left(\Gamma^{(1,n-2k)}_{a,t_n}\right)^{l_0} (g_a^{k+1}(t_n)),$$

with $i \geq 2$, and

$$\begin{aligned}
\Gamma^{(2,n-2k)}_{a,t_n} \circ \left(\Gamma^{(1,n-2k)}_{a,t_n}\right)^{l_0} (g_a^{k+1}(t_n)) < g_a^{k+1}(t_n) \quad &\Leftrightarrow \quad g_a^{k+2}(h^{l_0+1}_{a,t_n}(t_n)) < g_a^{k+1}(t_n) \\
&\Leftrightarrow \quad h^{l_0+1}_{a,t_n}(t_n) < g_a^{-1}(t_n).
\end{aligned}$$

Since

$$\Psi'_{a,t_n}(g_a^{k+1}(t_n)) = (g_a^{k+i})'(h^{l_0+1}(t_n))(h^{l_0}_{a,t_n})'(h_{a,t_n}(t_n))(g_a^{n-k})'(g_a^{k+1}(t_n)),$$

using the claim 2, (2.3.9), and the monotonicity of g_a, one gets

$$\begin{aligned}\Psi'_{a,t_n}(g_a^{k+1}(t_n)) &> (h_{a,t_n}^{l_0})'(h_{a,t_n}(t_n)) \\ &= (h_{a,t_n})'(h_{a,t_n}(t_n))(h_{a,t_n})'(h_{a,t_n}^2(t_n))\cdots(h_{a,t_n})'(h_{a,t_n}^{l_0}(t_n)) > 1\end{aligned}$$

To prove that

$$(\Psi_{a,t_n})'\left((\Gamma_{a,t_n}^{(1,n-2k)})^{-i}(g_a^{-1}(t_n))\right) > 1, \quad \text{for all } i = 1, \ldots, l_0,$$

from the monotonicity of g_a' on $[0, 1/2]$ it is enough to see that

$$(\Psi_{a,t_n})'\left((\Gamma_{a,t_n}^{(1,n-2k)})^{-l_0}(g_a^k(t_n))\right) > 1.$$

In fact

$$\begin{aligned}&(\Psi_{a,t_n})'\left((\Gamma_{a,t_n}^{(1,n-2k)})^{-l_0}(g_a^k(t_n))\right) \\ =\ &\left(\Gamma_{a,t_n}^{(2,n-2k)} \circ (\Gamma_{a,t_n}^{(1,n-2k)})^{l_0}\right)'\left((\Gamma_{a,t_n}^{(1,n-2k)})^{-l_0}(g_a^k(t_n))\right) \\ =\ &\left(g_a^{k+2} \circ h_{a,t_n}^{l_0-1} \circ g_{1,t} \circ g_a^{n-k}\right)'\left(g_a^{-n+k} \circ g_{1,t}^{-1} \circ h_{a,t_n}^{-l_0+1}(g_a^{-1}(t_n))\right) \\ =\ &\left(g_a^{k+2}\right)'(g_a^{-1}(t_n))\left(h_{a,t_n}^{l_0-1}\right)'\left(h_{a,t_n}^{-l_0+1}(g_a^{-1}(t_n))\right) \\ &\left(g_a^{n-k}\right)'(g_a^{-n+k} \circ g_{1,t}^{-1} \circ h_{a,t_n}^{-l_0+1}(g_a^{-1}(t_n)),\end{aligned}$$

and, once $g_a^{-n+k} \circ g_{1,t}^{-1} \circ h_{a,t_n}^{-l_0+1}(g_a^{-1}(t_n) < g_a^{k+1}(t_n)$, from the Claim 2, (2.3.9), and the monotonicity of g_a, we have

$$\begin{aligned}&(\Psi_{a,t_n})'\left((\Gamma_{a,t_n}^{(1,n-2k)})^{-l_0}(g_a^k(t_n))\right) \\ >\ &\left(g_a^{k+2}\right)'(g_a^{-1}(t_n))\left(h_{a,t_n}^{l_0-1}\right)'\left(h_{a,t_n}^{-l_0+1}(g_a^{-1}(t_n))\right)\left(g_a^{n-k}\right)'(g_a^{k+1}(t_n)) \\ =\ &\left(h_{a,t_n}^{l_0-1}\right)'\left((h_{a,t_n})^{-l_0+1}(g_a^{-1}(t_n))\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left(h_{a,t_n}^{l_0-1}\right)'\left((h_{a,t_n})^{-l_0+1}(g_a^{-1}(t_n))\right) &= h'_{a,t_n}(h_{a,t_n}^{-l_0+1}(g_a^{-1}(t_n)))\cdots h'_{a,t_n}(h_{a,t_n}^{-1}(g_a^{-1}(t_n))) \\
&= \frac{1}{h'_{a,t_n}(h_{a,t_n}^{l_0-1}(t_n))\cdots h'_{a,t_n}(h_{a,t_n}(t_n))} \\
&= \frac{1}{\left(h_{a,t_n}^{l_0-1}\right)'(h_{a,t_n}(t_n))}.
\end{aligned}$$

Since $h'_{a,t_n} = (g_a^{n+1})'$ is a decreasing map and $(h_{a,t_n})^{-l_0+1}\left(g_a^{-1}(t_n)\right) < h_{a,t_n}(t_n)$, we conclude that

$$\begin{aligned}
&\left(h_{a,t_n}^{l_0-1}\right)'\left((h_{a,t_n})^{-l_0+1}(g_a^{-1}(t_n))\right) > 1 \\
\Rightarrow\quad &(\Psi_{a,t_n})'\left((\Gamma_{a,t_n}^{(1,n-2k)})^{-l_0}(g_a^k(t_n))\right) > 1
\end{aligned}$$

and this finished the proof of the lemma. □

2.4 Heterodimensional cycles: a model family

In this section, we consider heterodimensional cycles. For that, in the model heterodimensional cycle introduced in Section 1.2, we will take the map F giving the central dynamics equal to $g_a : (-1/(2(e^a-1)), 2] \to \mathbb{R}$ such that:

$$g_a(y) = \frac{ye^a}{2ye^a + (1-2y)}.$$

As in Section 1.2 and for each $a > 0$, we have a one-parameter family of diffeomorphisms $(f_{a,t})_{t\in[-\tau,\tau]}$ unfolding a heterodimensional cycle at $t = 0$, associated to the fixed points $P = (0, 1/2, 0)$ and $Q = (0, 0, 0)$. We observe that

$$f_{a,0}(x, y, z) = f_a(x, y, z) = (\lambda_s x, g_a(y), \lambda_u z).$$

Recall that there is a small neighborhood $\mathcal{W}_a$ of the cycle, containing a neighborhood of the connection $\gamma = \{0\} \times (0, 1/2) \times \{0\}$ and the f_a-orbit of the heteroclinic point $(-1, 0, 0)$, that is a filtrating neighborhood of f_a. Let $\Lambda_{a,t} := \bigcap_{i\in\mathbb{Z}} f_{a,t}^i(\mathcal{W}_a)$ be the maximal $f_{a,t}$-invariant set in $\mathcal{W}_a$. As in the case of the family of skew-product maps $(G_{a,t})_{t\in[-1,1]}$, we have the following result.

Theorem 2.15. *Consider the arc of diffeomorphisms $(f_{a,t})_{t>0}$. The dynamics of $f_{a,t}$ in $\Lambda_{a,t}$ satisfies the following properties:*

(A) *For each $0 < a < \log 2$ there is $t_0 = t_0(a) > 0$ such that, for every $t \in (0, t_0]$,*

$$H(P, f_{a,t}) \subseteq H(Q, f_{a,t}) \text{ and } \Lambda_{a,t} = H(Q, f_{a,t}).$$

(B) *For $a \in (\log 2, \log 4)$ there are $t_0(a) > 0$, a sequence $t_n(a) \in (0, t_0(a)]$ converging to zero as $n \to +\infty$, and a sequence of intervals*

$$J(a, t_n) = [t_n(a) - \alpha_{a,t_n}, t_n(a) + \alpha_{a,t_n}]$$

such that $H(P, f_{a,t}) \subseteq H_{\mathcal{V}}(Q, f_{a,t}) = \Lambda_{a,t}$ for each $t \in J(a, t_n)$.

Using the remark 2.8 and applying Theorem 2.15 to $f_{a,t}^{-1}$ we obtain

$$H(Q, f_{a,t}^{-1}) \subseteq H(P, f_{a,t}^{-1}),$$

that is, $H(Q, f_{a,t}) \subseteq H(P, f_{a,t})$, then we can conclude the equality of the homoclinic classes

$$H(P, f_{a,t}) = H(Q, f_{a,t}),$$

for all $(a, t) \in (0, \log 2) \times (0, t_0(a)) \cup (\log 2, \log 4) \times J_{a,t_n}$.

Fix $t > 0$ small, $a > 0$ and denote by $D^P_{a,t} = [1/2 - t, g_a(1/2 - t)]$ the fundamental domain of g_a close to P. From (2.2.1), there is n large such that $t \in (t_{n+1}, t_n]$ and, as in Subsection 2.2, we consider $d_{a,t} = g_a^{-n}(1/2 - t)$ (see (2.2.7)). Thus, the corresponding fundamental domain of g_a close to Q, $D^Q_{a,t}$, satisfies

$$D^Q_{a,t} = [d_{a,t}, g_a(d_{a,t})] = g_a^{-k}([g_a^k(d_{a,t}), g_a^{k+1}(d_{a,t})]) = g_a^{-k}(D_{a,t}),$$

and, by construction, we have $g_a^n(D^Q_{a,t}) = D^P_{a,t}$. Now, as in Section 1.2.2, we consider the family of maps $\left(\Phi_{a,t}^{(q,p)}\right)_{q,p\geq 0}$, defined by

$$\Phi_{a,t}^{q,p} : D_{a,t}^{q,p} \to (d_{a,t}, g_a(d_{a,t})] , \quad y \mapsto g_a^q\left(g_a^{n+p}(y) - \frac{1}{2} + t\right),$$

with $D^{q,p}_{a,t} = \{y \in D^Q_{a,t} : \Phi^{q,p}_{a,t}(y) \in D^Q_{a,t}\}$, which describe the dynamics of $f_{a,t}$ in the central direction and the following equality

$$\Phi^{q,p}_{a,t} = g_a^{-k} \circ \Gamma^{(q,n-2k+p)}_{a,t} \circ g_a^k \tag{2.4.1}$$

holds.

To get the inclusion $H(Q, f_{a,t}) \subset H(P, f_{a,t})$, it is enough to see that the two-parameter family $\mathfrak{F}_{a,t}$ defined by

$$\mathfrak{F}_{a,t} = \{\Phi^{q,p}_{a,t} : (q,p) \in \mathbb{N} \times \mathbb{N}\}$$

of iterated function systems satisfies the following expansiveness property:

Expansion condition for heterodimensional cycles (EC'): *There are* $l_{a,t} > 1$ *and* $r_a \in \mathbb{N}$ *such that for every* $y \in (d_{a,t}, g_a(d_{a,t})]$ *there is a* l*-block*

$$\varrho_l = \varrho_l(y) = [(m_1, n_1), \dots, (m_l, n_l)],$$

with $l \leq r_a$*, satisfying* $\left(\Phi^{\varrho_l}_{a,t}\right)'(y) \geq l_{a,t}$.

First it is important to note that, from (2.4.1) and Lemma 2.4, the system $\mathfrak{F}_{a,t}$ satisfies the condition **(EC')** for each $a \in (0, \log 2)$ and $t \in (0, t_0(a)]$, where $t_0(a)$ is defined in the proof of Proposition 2.5. From Proposition 2.14, the system of iterated functions $\mathfrak{F}_{a,t}$ also satisfies **(EC')**, for each $a \in (\log 2, \log 4)$ and $t \in J(a, t_n)$, see (2.3.3) to recall the definition of the intervals $J(a, t_n)$.

Now, the main step towards proving Theorem 2.15 is presented in the next lemma.

Lemma 2.16. *Consider a small* t *and an open interval* $J \subset (d_{a,t}, g_a(d_{a,t})]$*. Under the condition* **(EC')***, there are* $i_1, \dots, i_j \in \mathbb{N}$*, with* $j \in \mathbb{N}$*, and a sequence of blocks* $(\varrho_{i_1}, \dots, \varrho_{i_j})$ *such that*

$$d_{a,t} \in \Phi^{\varrho_{i_j}}_{a,t} \circ \cdots \circ \Phi^{\varrho_{i_1}}_{a,t}(J).$$

In particular, there is $y \in J$ *such that* $h_{a,t} \circ \Phi^{\varrho_{i_j}}_{a,t} \circ \cdots \circ \Phi^{\varrho_{i_1}}_{a,t}(y) = 0$.

Proof. Writing $J = [a_0, b_0]$, from condition **(EC')**, there are $i_1 \in \mathbb{N}$ and a i_1-block,

$\varrho_{i_1} = [(q_1, p_1), \dots, (q_{i_1}, p_1)]$, such that $\left(\Phi_{a,t}^{\varrho_{i_1}}\right)'(b_0) \geq l_{a,t} > 1$. Thus, either

$$d_{a,t} \in \Phi_{a,t}^{q_i,p_1} \circ \cdots \circ \Phi_{a,t}^{q_1,p_1}(J)$$

for some $i \in \{1, \dots, i_1\}$ or $|\Phi_{a,t}^{\varrho_{i_1}}(J)| \geq l_{a,t}|J|$.

Proceeding in same way with $\Phi_{a,t}^{\varrho_{i_1}}(J)$ and so on, we obtain

$$|\Phi_{a,t}^{\varrho_{i_j}} \circ \cdots \circ \Phi_{a,t}^{\varrho_{i_1}}(J)| \geq l_{a,t}^j|J|, \quad l_{a,t} > 1. \tag{2.4.2}$$

Consequently there is a m-block $\varrho_m^\star$ such that $d_{a,t} \in \Phi_{a,t}^{\varrho_m^\star}(J)$ and therefore there is $y \in J$ such that $h_{a,t}(\Phi_{a,t}^{\varrho_m^\star}(y)) = h_{a,t}(d_{a,t}) = 0$. □

Now, as in [DS04, Theorem 2.1], the proof of the Theorem 2.15 follows from the Lemma 2.16 and we omit it here. However, it is relevant to refer that the prove's key step is to show that the stable manifold of Q intersects any disk Δ transverse to the stable manifold of P. If the condition **(EC')** holds, one has that these discs have a return to $\mathcal{C}$ by a power of f_t such that their "central size" increases exponentially.

Remark 2.17. *The conclusion of Theorem 2.5 remains valid for the diffeomorphisms f_t, introduced in Subsection 1.2.1, verifying the condition* **(EC')**.

Finally we observe that in the case of heterodimensional cycles, assuming the condition **(EC')**, one has

$$W^u(P, f_t) \cap W^s(Q, f_t) \subset H(Q, f_t)$$

and, for the one-parameter family of skew-product maps $(G_t)_{t\geq 0}$ introduced in Section 1.3, we claim that

$$\Big(W^u(P_t, G_t) \cap W^s(Q_t, G_t)\Big) \cap H_{\mathcal{V}}(Q_t, G_t) = \emptyset. \tag{2.4.3}$$

In fact, in the case of heterodimensional cycles, to get the inclusion

$$W^u(P, f_t) \cap W^s(Q, f_t) \subset H(Q, f_t),$$

it is enough to construct, for every $w \in W^u(P, f_t) \cap W^s(Q, f_t)$, a sequence $(w_n)_n$ in $H(Q, f_t)$ such that $w_n \to w$, as $n \to \infty$. This was proved in [DS04]. Recall that, for w_n close to w, it is possible that the itineraries of these two points are different while, in the case of skew-product maps, this cannot happen that is, if $A_n = (\xi^n, x_n)$ is close to $A = (\xi, x)$ then x_n is close to x, ξ_i^n is equal to ξ_i for a very long time, and the sequence ξ give the itinerary of the point x.

To prove the claim (see (2.4.3)), consider

$$X = (\xi, x) \in W^u(P, G_t) \cap W^s(Q, G_t);$$

we need to show that $X \notin \overline{W^s(Q_t, G_t) \pitchfork W^u(Q_t, G_t)}$. Without loss of generality we can assume that

$$X = (0^{-\mathbb{N}}.\omega_0 \cdots \omega_{k_0}\xi_{k_0+1} \cdots \xi_m 0^{\mathbb{N}}, p_t),$$

with $g_{[\omega_0 \cdots \omega_{k_0}\xi_{k_0+1} \cdots \xi_m],t}(p_t) = q_t$.

First we prove that $X \notin W^s(Q_t, G_t) \cap W^u(Q_t, G_t)$. Since Q_t is a fixed point of expanding type, it is enough to prove that, for all open interval I containing p_t, $\xi \times I \nsubseteq W^u(Q_t, G_t)$. In fact, for each open interval I, there is $\rho > 0$ small such that $I \supseteq (p_t - \rho, p_t + \rho)$ and, as $\xi_{-i} = 0$ for $i \in \mathbb{N}$, we have

$$g_{0,t}^{-i}([p_t, p_t + \rho)) \cap (q_t, p_t) = \emptyset, \text{ for every } i \in \mathbb{N}.$$

Now we prove that X cannot be accumulated by homoclinic points of Q. Let $\delta > 0$, with $\delta \ll 2^{-(m+2)}$, and suppose that there is $Y = (\zeta, y) \in \Lambda_t$ such that $Y \in W^s(Q_t, G_t) \pitchfork W^u(Q_t, G_t)$ and $d(X, Y) < \delta$. Consequently we have $d(y, p_t) < \delta$, $d_{\Sigma_2}(\xi, \zeta) < \delta$, and, from the definition of y, $y < p_t$ which implies that

$$g_{[\omega_0 \cdots \omega_{k_0}\xi_{k_0+1} \cdots \xi_m],t}(y) < g_{[\omega_0 \cdots \omega_{k_0}\xi_{k_0+1} \cdots \xi_m],t}(p_t) = q_t$$

that is $g_{[\omega_0 \cdots \omega_{k_0}\xi_{k_0+1} \cdots \xi_m],t}(y) < q_t$, thus $G_t^i(Y) \notin \mathcal{V}$, for some $i \in \mathbb{N}$, this contradicts the fact that $Y \in \Lambda_t$ and the claim is proved.

Chapter 3

Non-hyperbolic homoclinic classes - a more general case

In this chapter, we study the dynamics of the one-parameter family of skew-product maps $G_t : \Sigma_n \times \mathbb{K} \to \Sigma_n \times \mathbb{K}$, $t \in [-1, 1]$ introduced in Section 1.3, with $\mathbb{K} = [-1, 1]$ or $\mathbb{K} = S^1$. First we prove that there are at most two homoclinic classes. Afterwards, we present a sufficient condition for obtaining $H_{\mathcal{V}}(Q, G_t) \subseteq H_{\mathcal{V}}(P, G_t)$ and finally we prove that the growth of the number of periodic orbits in Λ_t of the family $(G_t)_{t\geq 0}$ is not super-exponential. The same properties are also studied for the one-parameter family $(f_t)_{t\in[-1,1]}$, where t is related to the unfolding of the cycle, defined in Section 1.2.

3.1 Number of homoclinic classes

In this section we prove that the set of periodic points whose orbit is contained in $\mathcal{V}$ is a subset of $H_{\mathcal{V}}(P, G_t) \cup H_{\mathcal{V}}(Q, G_t)$.

Recall the definition of G_t in section 1.3. Until further notice we assume that the transition map $g_{[\omega_0\cdots\omega_{k_0}],0}$ (see (1.3.12)) preserves the orientation in a small neighborhood of $1/2$.

Since g_0' is strictly decreasing on $[0, 1/2]$, choosing $\gamma > 0$ sufficiently small in the definition of the neighborhood of the cycle (see (1.3.9)) and k sufficiently large, we can assume that the derivative of $g_{[\omega_0\cdots\omega_{k_0}],0} = g_{[0^k\alpha_0\cdots\alpha_r 0^k],0}$ is strictly decreasing in a neighborhood of $1/2$.

Recall that, for each $t > 0$ sufficiently small, $g_{0,t}$ preserves the orientation and has two fixed points, an attractor p_t close to $1/2$ and a repeller q_t close to 0. We also have that $\mathbf{g'_{0,t}}$ **is strictly decreasing on** $[q_t, p_t]$.

Proposition 3.1. *For every $t > 0$ small enough, the periodic points of G_t in Λ_t are contained in $H_{\mathcal{V}}(P_t, G_t) \cup H_{\mathcal{V}}(Q_t, G_t)$.*

Proof. Let us choose and fix $t > 0$ small such that $g_{[\omega_0 \cdots \omega_{k_0}],t}$ preserves the orientation on $\mathbb{K}$ and $\left(g_{[\omega_0 \cdots \omega_{k_0}],t}\right)'$ is strictly decreasing in a neighborhood of p_t. Consider a periodic point $X = \left((\eta_0 \cdots \eta_{l-1})^{\mathbb{Z}}, x\right) \in \Lambda_t$ of G_t. There is no loss of generality in assuming that $X \in \Delta_t = [0^{-k}.0^k] \times D_t$ where $D_t = [d_t, g_{0,t}(d_t)]$ is a fundamental domain of $g_{0,t}$ and $d_t \in (g_{[\omega_0 \cdots \omega_{k_0}],t}(p_t), p_t)$ is such that $g_{[0^h \omega_0 \cdots \omega_{k_0}],t}(d_t) = q_t$, for some $h = h(d_t) \in \mathbb{N}$.

First we assume that X is of expanding type, i.e.,

$$g_{[\eta_0 \cdots \eta_{l-1}],t}(x) = x \text{ and } \left(g_{[\eta_0 \cdots \eta_{l-1}],t}\right)'(x) > 1.$$

The goal is to prove that in this case $X \in H_{\mathcal{V}}(Q, G_t)$.

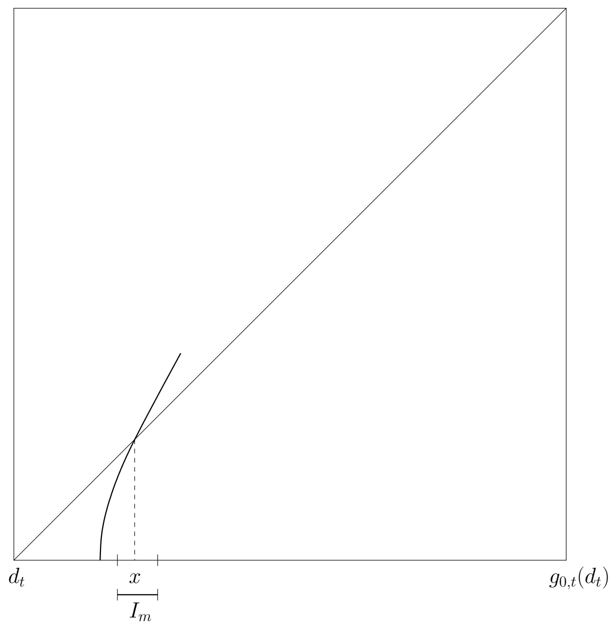

Figure 3.1: The expanding point $X = \left((\eta_0 \cdots \eta_{l-1})^{\mathbb{Z}}, x\right)$

From the monotonicity of $g_{0,t}$ and $g'_{0,t}$ on $[q_t, p_t]$ and noting that $g_{[\omega_0 \cdots \omega_{k_0}],t}$ preserves the orientation in a neighborhood of p_t and X is of expanding type, it follows that there is $j(m) \in \mathbb{N}$ (see Figure 3.1) such that

$$d_t \in \left(g_{[\eta_0 \cdots \eta_{l-1}],t}\right)^{j(m)}(I_m),$$

where $I_m := [x - 1/m, x + 1/m]$, that is, there is $x_m \in I_m$ such that

$$\left(g_{[\eta_0 \cdots \eta_{l-1}],t}\right)^{j(m)}(x_m) = d_t.$$

Note that $j(m) \to \infty$, as $m \to \infty$. Using the definition of d_t, we get

$$X_m = \left(0^{-\mathbb{N}}(\eta_0 \cdots \eta_{l-1})^{-m}.(\eta_0 \cdots \eta_{l-1})^{j(m)} 0^h \omega_0 \cdots \omega_{k_0} 0^{\mathbb{N}}, x_m\right) \in W^s(Q_t, G_t).$$

Moreover, by construction, the point X_m is a transverse homoclinic point of Q. Since $X_m \to X$, as $m \to \infty$, we conclude that $X \in H_{\mathcal{V}}(Q, G_t)$.

Since the same arguments remain valid for $X = (\xi, x)$ with

$$g_{[\eta_0 \cdots \eta_{l-1}],t}(x) = x \text{ and } \left(g_{[\eta_0 \cdots \eta_{l-1}],t}\right)'(x) = 1,$$

we also have, in this case, that $X \in H(Q, G_t)$. Indeed, for all $m \in \mathbb{N}$ large, there is $j(m) \in \mathbb{N}$ such that $d_t \in \left(g_{[\eta_0 \cdots \eta_{l-1}],t}\right)^{j(m)}([x - 1/m, x + 1/m])$. Similarly one has $j(m) \to \infty$ as $m \to \infty$.

Now we assume that X is of contracting type, that is $\left(g_{[\eta_0 \cdots \eta_{l-1}],t}\right)'(x) < 1$. Considering

$$A_t^\eta := \{x \in D_t : g_{[\eta_0 \cdots \eta_{l-1}],t}(x) \in D_t\},$$

then two possibilities can happen: either $A_t^\eta = D_t$ or $A_t^\eta \subsetneq D_t$.

Case 1. $A_t^\eta = D_t$

By the definition of D_t, there is $u_0 \in \mathbb{N}$ such that

$$g_{[\omega_0 \cdots \omega_{k_0} 0^{u_0}],t}(p_t) \in D_t. \tag{3.1.1}$$

Writing $\tilde{p}_t := g_{[\omega_0 \cdots \omega_{k_0} 0^{u_0}],t}(p_t)$, for $m \in \mathbb{N}$ large, there is $j(m) \in \mathbb{N}$ (see Figure 3.2) such that

$$g^{j(m)}_{[\eta_0 \cdots \eta_{l-1}],t}(\tilde{p}_t) \in I_m,$$

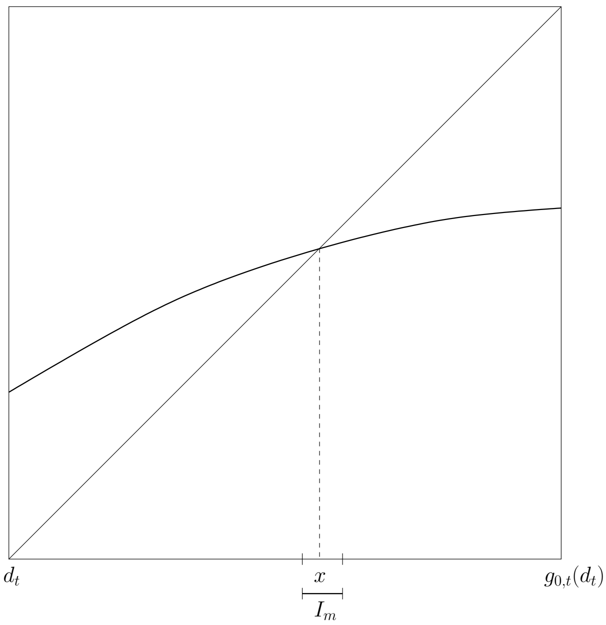

Figure 3.2: The contracting point $X = \big((\eta_0 \cdots \eta_{l-1})^{\mathbb{Z}}, x\big)$ in case 1.

where $I_m := [x - 1/m, x + 1/m]$. Thus $g^{j(m)}_{[\eta_0 \cdots \eta_{l-1}],t}(\tilde{p}_t) = x'_m$ for some $x'_m \in I_m$. Therefore the point

$$X_m = \big(0^{-\mathbb{N}} \omega_0 \cdots \omega_{k_0} 0^{-u_0} (\eta_0 \cdots \eta_{l-1})^{-j(m)}.(\eta_0 \cdots \eta_{l-1})^m 0^{\mathbb{N}}, x'_m\big)$$

is a homoclinic point of P with $X_m \to X$, as $m \to \infty$, for which we conclude $X \in H_{\mathcal{V}}(P, G_t)$.

Case 2. $A^{\eta}_t \subsetneq D_t$

In this case, since $g_{0,t}$ and $g_{[\omega_0 \cdots \omega_{k_0}],t}$ preserve the orientation, the map $g_{[\eta_0 \cdots \eta_{l-1}],t}$ also preserves the orientation and its graphic is below the diagonal in the interval $[x, p_t]$ (see Figure 3.3). Note that $(g_{0,t}(d_t), g^2_{0,t}(d_t)]$ is also a fundamental domain in $g_{0,t}$ and

$$g_{0,t}(\tilde{p}_t) = g_{[\omega_0 \cdots \omega_{k_0} 0^{u_0+1}],t}(p_t) \in (g_{0,t}(d_t), g^2_{0,t}(d_t)].$$

Recall the definition of u_0 in (3.1.1). Since $g'_{0,t}$ is strictly decreasing on $[q_t, p_t]$ and $g_{[\omega_0 \cdots \omega_{k_0}],t}$ is close to a affine map in a neighborhood of p_t, we have

$$\left(g_{[\eta_0 \cdots \eta_{l-1}],t}\right)^j (g_{0,t}(\tilde{p}_t)) \to x, \quad \text{as } j \to +\infty,$$

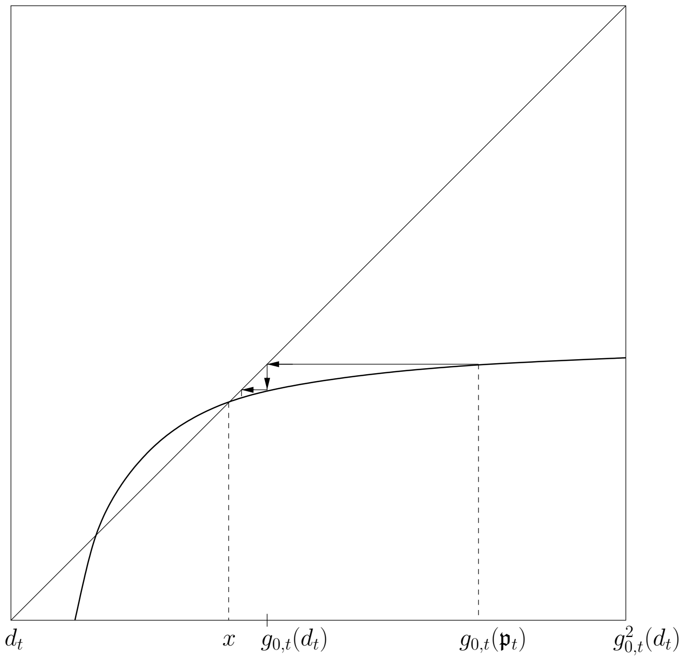

Figure 3.3: The contracting point $X = \big((\eta_0 \cdots \eta_{l-1})^{\mathbb{Z}}, x\big)$ in case 2.

and, for every large $m \in \mathbb{N}$, there are $j(m)$ and $x_m \in [x - 1/m, x + 1/m]$, such that

$$\left(g_{[\eta_0 \cdots \eta_{l-1}],t}\right)^{j(m)} (g_{0,t}(\tilde{p}_t)) = x_m.$$

Consequently,

$$X_m = (0^{-\mathbb{N}} \omega_0 \cdots \omega_{k_0} 0^{-(u_0+1)} (\eta_0 \cdots \eta_{l-1})^{-j(m)}.(\eta_0 \cdots \eta_{l-1})^m 0^{\mathbb{N}}, x_m) \in H_{\mathcal{V}}(P, G_t),$$

which proves the proposition. □

The next goal is to prove a similar result for heterodimensional cycles introduced in Section 1.2. Recall the definition of f_t in Section 1.2 and note that the maps

$$\Phi_t^{m,n} : D_t^{m,n} \to D_Q^t, \quad y \mapsto F^m \circ F_{1,t} \circ F^{n+N_t}(y),$$

where $F_{1,t}(y) = y - 1/2 + t$ in a neighborhood of $1/2$, describes the dynamics of f_t in the central direction. Recall that F is a strictly increasing function with F' strictly decreasing and it has two fixed points, 0 and $1/2$, such that $F'(0) = \beta > 1$ and $F'(1/2) = \lambda < 1$.

Moreover, f_t has a heterodimensional cycle at $t = 0$ associated to the fixed saddles of $P = (0, 1/2, 0)$ and $Q = (0, 0, 0)$ of indices one and two, respectively. We also have that

$$\begin{aligned} W^s(P, f_t) &\supseteq [-2,2] \times \left(0, \tfrac{1}{2}\right] \times \{0\} \\ W^s(Q, f_t) &\supseteq [-2,2] \times \{0\} \times \{0\} \\ W^u(P, f_t) &\supseteq \{0\} \times \{\tfrac{1}{2}\} \times [-2,2] \\ W^u(Q, f_t) &\supseteq \{0\} \times [0, \tfrac{1}{2}) \times [-2,2] \end{aligned}$$

and, for $t > 0$, $X_t = (-1, t, 0)$ is a transverse homoclinic point of P and $Y_t = (-1, 0, 0)$ is a transverse homoclinic point of Q for f_t (see Figures 1.1 and 1.3).

Consider a filtrating neighborhood $\mathcal{W}$ of the heterodimensional cycle and let Λ_t be the maximal invariant of f_t in $\mathcal{W}$. As in the case of the skew-product maps G_t, we have the following proposition:

Proposition 3.2. *For every $t > 0$ small enough, the set of periodic points of f_t in Λ_t is contained in $H(P, f_t) \cup H(Q, f_t)$.*

Proof. Let $A = (x, y, z) \in \Lambda_t$ be a periodic point of f_t. Replacing A by some iterate, we can assume that $y \in D_t^Q = [d_t, F(d_t)]$ and, from the periodicity of A, there is a l-block $\varrho_l = [(m_1, n_1), \ldots, (m_l, n_l)]$ such that

$$\Phi_t^{\varrho_l}(y) = \Phi_t^{m_l, n_l} \circ \cdots \circ \Phi_t^{m_1, n_1}(y) = y,$$

where $l \in \mathbb{N}$ is minimal with this property. Therefore the period of A is

$$l(N_t + k_0) + \sum_{i=1}^{l} m_i + \sum_{i=1}^{l} n_i,$$

where k_0 is as in the definition of the cycle (see (1.2.2)) and $F^{N_t}(D_t^Q) = D_t^P$.

First we assume that

$$(\Phi_t^{\varrho_l})'(y) > 1,$$

i.e. the index (dimension of the unstable bundle) of A is 2.

Once f_t^{-1} expands exponentially in the $\mathbb{X}$-direction, one has

$$[x - \epsilon, x + \epsilon] \times \{(y, z)\} \pitchfork W^u(Q, f_t) \neq \emptyset, \text{ for every } \epsilon > 0,$$

which implies that there is a sequence $A_n = (x_n, y, z) \in W^u(Q, f_t)$ with $A_n \to A$, as $n \to +\infty$.

We claim that, for all $\delta > 0$, the set

$$U(\delta) = \{x_n\} \times [y - \delta, y + \delta] \times [z - \delta, z + \delta] \subset W^u(Q, f_t)$$

intersects $W^s(Q, f_t)$ transversally, where x_n is such that $A_n = (x_n, y, z) \in W^u(Q, f_t)$. Using the filtrating neighborhood and the local structure in the neighborhood of the cycle, it suffices to show that there is $k(\delta) \in \mathbb{N}$ such that

$$f_t^{k(\delta)}(U(\delta)) \pitchfork ([-1, 1] \times \{0\} \times \{0\}).$$

Since $\Phi_t^{\varrho_l}(y) = y$, $(\Phi_t^{\varrho_l})'(y) > 1$ and $(\Phi_t^{\varrho_l})'$ is a decreasing map (see Figure 3.1), there are $j = j(\delta)$ (by shrinking δ if necessary) and $\tilde{y} \in (y - \delta, y)$ such that

$$(\Phi_t^{\varrho_l})^j(\tilde{y}) = d_t,$$

and, recalling that $h_t = F_{1,t} \circ F^{N_t}$ (see (1.2.3)), we conclude that $h_t \circ (\Phi_t^{\varrho_l})^j(\tilde{y}) = 0$.

Observe that f_t expands exponentially in the $\mathbb{Z}-$ direction, so, by the configuration of the cycle, we conclude that there is $k(\delta) > l(j+1)(k_0 + N_t) + j\sum_{i=1}^{l}(m_i + n_i)$ such that $f_t^{k(\delta)}(U(\delta))$ contains a disc $U^\star(\delta)$ of the form

$$U^\star(\delta) = \{x_n^\star\} \times [y_1, y_2] \times [-2, 2], \; x_n^\star \in [-2, 2], \; y_1 < 0 < y_2.$$

Hence, since the interior of $U^\star(\delta)$ is contained in $W^u(Q, f_t)$ and $U^\star(\delta)$ meets $W^s(Q, f_t)$ transversely, $U^\star(\delta)$ contains a homoclinic point A_δ^n of Q. Letting $\delta \to 0$, we get $A_\delta^n \to A_n$, which implies that $A_n \in H(Q, f_t)$, and thus $A \in H(Q, f_t)$.

As in Proposition 3.1, the same arguments remains valid if $(\Phi_t^{\varrho_l})'(y) = 1$, and we get $A \in H(Q, f_t)$.

Now we assume that $(\Phi_t^{\varrho_l})'(y) < 1$, that is, the index of A is one. Since f_t expands exponentially in the $\mathbb{Z}$- direction, we get

$$\{(x, y)\} \times [z - \epsilon, z + \epsilon] \pitchfork W^s(P, f_t) \neq \emptyset,$$

then A is accumulated by points $\tilde{A}_n = (x, y, z_n) \in W^s(P, f_t)$. The next goal is to prove that $\tilde{A}_n$ is accumulated by homoclinic points of P.

For $\delta > 0$ small, consider the rectangle

$$S(\delta) = (x - \delta, x + \delta) \times (y, y + \delta) \times \{z_n\} \subset W^s(P, f_t).$$

Using the properties of F and $F_{1,t}$, we conclude that the graph of $\Phi_t^{\varrho_l}$ is below the diagonal on the interval $[y, F(d_t)]$. Since $F(t) \in D_t^Q = [d_t, F(d_t)]$, one has $F^2(t) > F(d_t)$, then there is a first $j_0 = j_0(\delta) \in \mathbb{N}$ such that

$$(\Phi_t^{\varrho_l})^j \circ F^2(t) \in (y, y + \delta), \ \forall j \geq j_0.$$

Thus, by the configuration of the cycle, the semi-local product structure, and noting that f_t^{-1} expands in the $\mathbb{X}$-direction, there is $j = j(\delta)$ such that for

$$k(\delta) = jl(k_0 + N_t) + j \sum_{i=1}^{l} (m_i + n_i) + 2,$$

$f_t^{-k(\delta)}(S(\delta))$ contains a disc $S^\star$ of the form

$$S^\star(\delta) = [-2, 2] \times (l_1, l_2) \times \overline{z_n}, \text{ with } l_1 < t < l_2$$

Consequently $S^\star(\delta)$ meets $W^u(P, f_t)$ transversely, and so $S^\star(\delta)$ contains a homoclinic point of P and the same holds for $S(\delta)$, ending the proof of the proposition. □

As the periodic points are dense in the homoclinic class, from Proposition 3.1 (Proposition 3.2), it follows that $G_t|_{\Lambda_t}$ $(f_t|_{\Lambda_t})$ has at most two homoclinic classes.

3.2 Dense orbits for the system of iterated functions

In this section we will construct a set $\Re_t$ of homoclinic points related to P_t, whose orbits are contained in $\mathcal{V}$, of the form (ξ, y), such that $\{y : (\xi, y) \in \Re_t\}$ is dense in a fundamental domain of $g_{0,t}$.

Let $t > 0$ be sufficiently small and write $\mathfrak{p}_t := g_{[\omega_0 \cdots \omega_{k_0}],t}(p_t)$. In what follows, we consider the fundamental domains of $g_{0,t}$,

$$\mathfrak{D}_t^1 := \left(g_{0,t}^{-2}(\mathfrak{p}_t), g_a^{-1}(\mathfrak{p}_t)\right] \text{ and } \mathfrak{D}_t^0 := \left(g_{0,t}^{-1}(\mathfrak{p}_t), \mathfrak{p}_t\right],$$

and the interval $\mathfrak{D}_t := \mathfrak{D}_t^0 \cup \mathfrak{D}_t^1 = \left(g_{0,t}^{-2}(\mathfrak{p}_t), \mathfrak{p}_t\right]$.

Let j_0 be the first $j \in \mathbb{N}$ such that $g_{0,t}^j\left(g_{0,t}^{-2}(\mathfrak{p}_t)\right) \in [g_{[\omega_0\cdots\omega_{k_0}\cdot],t}(q_t), p_t]$ (see (1.3.2) for the definition of $g_{[\omega_0\cdots\omega_{k_0}\cdot],t}$) and, for each $j > j_0$, define the map

$$\Gamma_t^{(0,j)} : \mathfrak{D}_t \to (q_t, \mathfrak{p}_t], \quad \Gamma_t^{(0,j)}(x) = g_{[0^j\omega_0\cdots\omega_{k_0}],t}(x).$$

Consequently, for j large, we have $\Gamma_t^{(0,j)}(\mathfrak{D}_t) \subseteq \mathfrak{D}_t$ and, once the maps $g_{0,t}$ and $g_{\omega_0\cdots\omega_{k_0}}$ are order preserving, there is $j_1 > j_0$ such that

$$\Gamma_t^{(0,j)}(\mathfrak{D}_t) \subseteq \mathfrak{D}_t, \quad \text{for all } j \geq j_1. \tag{3.2.1}$$

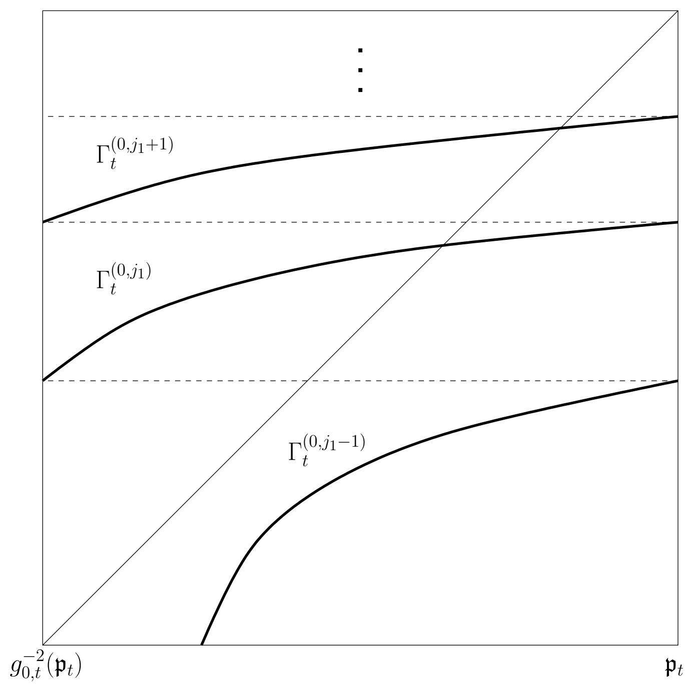

Figure 3.4: The maps $\Gamma_t^{(0,j)}$

The goal of this section is to prove the following theorem.

Theorem 3.3. *Assume that there is $j^\star \geq j_1$ such that the map $\Gamma_t^{(0,j^\star)}$ has an attractive fixed point $s_{j^\star} \in \left(g_{0,t}^{-2}(\mathfrak{p}_t), g_{0,t}^{-1}(\mathfrak{p}_t)\right)$. Then $H_{\mathcal{V}}(Q_t, G_t) \subseteq H_{\mathcal{V}}(P_t, G_t)$.*

Before present the proof and some preliminary results, it is convenient to observe that the hypothesis of Theorem 3.3 holds for a skew-product map G_t with $\lambda = (g_{0,t})'(p_t)$ close to 1 .

In the same way, for the family $(G_{a,t})_{t\geq 0}$ introduced in the previous chapter, we have

$$\mathfrak{p}_t = g_{[\omega_0\cdots\omega_{k_0}],t}(1/2) = g_{[0^k10^k],t}(1/2) = g_a^k(t)$$

and, for $a < \log\big((1+\sqrt{5})/2\big)$, there are $j^\star = j^\star(a)$ ($j^\star$ minimal with this property) and $n_0 = n_0(a)$ such that for each $n \geq n_0$ and $t \in \Big[t_n(1-\mu^-(a,t_n)), t_n(1+\mu^+(a,t_n))\Big)$, there is a fixed point $s_{j^\star}$ of $\Gamma_{a,t}^{(0,j^\star)}$ on $\mathfrak{D}_{a,t}^1 = (g_a^{k-2}(t), g_a^{k-1}(t)]$. Thus the hypothesis of Theorem 3.3 also holds in this case. We observe that

$$\lim_{n\to+\infty}\big(1-\mu^-(a,t_n)\big) = \frac{e^{-\frac{(j^\star-3)}{2}a}\sqrt{(e^{2a}-1)}}{e^{2a}-1}$$

and

$$\lim_{n\to+\infty}\big(1+\mu^+(a,t_n)\big) = \frac{e^{-\frac{(j^\star-1)a}{2}}\sqrt{(e^{a}-1)}}{e^{a}-1}.$$

In the next proposition we find a condition for a such that, for all $t \in (t_{n+1}, t_n]$ small enough, we have a attractive fixed point between $g_a^{k-2}(t)$ and $g_a^{k-1}(t)$.

Proposition 3.4. *For* $0 < a < \log\left((\sqrt[3]{(27-3\sqrt{69})/2})/3 + \sqrt[3]{(9+\sqrt{69})/18}\right)$*, there are* $j^\star \in \mathbb{N}$ *and* $t_0 = t_0(a)$ *such that, for each* $t \in (0, t_0)$*, the map* $\Gamma_t^{(0,j^\star)}$ *has an attractive fixed point* $s_{j^\star} \in \big(g_{0,t}^{k-2}(t), g_{0,t}^{k-1}(t)\big)$*. Moreover, one has* $j^\star > 4$*.*

Proof. First we claim that, for $0 < a < \log\left((\sqrt[3]{(27-3\sqrt{69})/2})/3 + \sqrt[3]{(9+\sqrt{69})/18}\right)$, there is $t_0 > 0$, such that for all $t \in (0, t_0)$, if $s_{j+1} \geq g_a^{k-1}(t)$ then $s_j > g_a^{k-2}(t)$, where s_j is an attractive fixed point of $\Gamma_{a,t}^{(0,j)}$.

Note that the condition $s_{j+1} \geq g_a^{k-1}(t)$ is equivalent to $\Gamma_{a,t}^{(0,j+1)}(g_a^{k-1}(t)) \geq g_a^{k-1}(t)$ and $s_j > g_a^{k-2}(t)$ is equivalent to $\Gamma_{a,t}^{(0,j)}(g_a^{k-2}(t)) > g_a^{k-2}(t)$.

Writing $t = t_n(1+\mu)$, the inequalities

$$\begin{cases} \displaystyle\lim_{n\to+\infty} \frac{\Gamma_{a,t}^{(0,j+1)}(g_a^{k-1}(t_n(1+\mu))) - g_a^{k-1}(t_n(1+\mu))}{t_n} & \geq & 0 \\[2ex] \displaystyle\lim_{n\to+\infty} \frac{\Gamma_{a,t}^{(0,j)}(g_a^{k-2}(t_n(1+\mu))) - g_a^{k-2}(t_n(1+\mu))}{t_n} & > & 0 \end{cases}$$

are equivalent to

$$\begin{cases} 1+\mu & \geq \dfrac{e^{-\frac{j}{2}a}\sqrt{(e^a-1)}}{e^a-1} \\ \\ 1+\mu & > \dfrac{e^{-\frac{(j-3)}{2}a}\sqrt{(e^{2a}-1)}}{e^{2a}-1} \end{cases},$$

thus we just need to prove that, for $0 < a < \log\left(\frac{1}{3}\sqrt[3]{\frac{27-3\sqrt{69}}{2}} + \sqrt[3]{\frac{9+\sqrt{69}}{18}}\right)$,

$$\frac{e^{-\frac{(j-3)}{2}a}\sqrt{(e^{2a}-1)}}{e^{2a}-1} < \frac{e^{-\frac{j}{2}a}\sqrt{(e^a-1)}}{e^a-1}.$$

In fact

$$\begin{aligned} \frac{e^{-\frac{(j-3)}{2}a}\sqrt{e^{2a}-1}}{e^{2a}-1} < \frac{e^{-\frac{j}{2}a}\sqrt{(e^a-1)}}{e^a-1} \quad & \Leftrightarrow \quad \frac{e^{-(\frac{j-3}{2})a}\sqrt{e^{2a}-1}}{e^{-\frac{j}{2}a}\sqrt{e^a-1}} < \frac{e^{2a}-1}{e^a-1} \\ & \Leftrightarrow \quad e^{3a}-e^a-1<0 \\ & \Leftrightarrow \quad a < \log\left(\frac{1}{3}\sqrt[3]{\frac{27-3\sqrt{69}}{2}} + \sqrt[3]{\frac{9+\sqrt{69}}{18}}\right), \end{aligned}$$

which concludes the proof of the claim.

For the second part of the proposition, note that

$$\lim_{n\to+\infty} \frac{\Gamma^{(0,4)}_{a,t}(g_a^{k-2}(t_n)) - g_a^{k-2}(t_n)}{t_n} = 1 - e^{-3a} - e^{-2a}.$$

and $1-e^{-3a}-e^{-2a} < 0$ for $a < \log\left((\sqrt[3]{(27-3\sqrt{69})/2})/3 + \sqrt[3]{(9+\sqrt{69})/18}\right)$. Therefore $j^\star \geq j_1 > 4$ (see (3.2.1) for the definition of j_1) and this finish the proof of the proposition. □

We observe that, in the previous chapter, we conclude that

$$H_{\mathcal{V}}(Q, G_{a,t}) = H_{\mathcal{V}}(P, G_{a,t}),$$

for $a < \log 2$ and t sufficiently small, using symmetric properties of g_a. Moreover we did not use the hypothesis of Theorem 3.3.

To prove Theorem 3.3, we first consider a skew-product map G_t for which the hypothesis holds and we construct multisequences contained in $\mathfrak{G}_t$-orbit of $\mathfrak{p}_t$ verifying the properties listed in Proposition 3.5. To construct the multisequences we follow the approach suggested in [DHRS09]. Then arguing as in [D95b, Lemma 4.1] we conclude that the closure of the points of the sequences contains $\mathfrak{D}_t^0$. Finally, using this fact, we prove that $H_{\mathcal{V}}(Q_t, G_t) \subseteq H(P_t, G_t)$.

We denote by $[\mathfrak{b}]_n$ a n-tuple of natural numbers $[\mathfrak{b}]_n = i_1, \cdots, i_n$ with $i_j \in \mathbb{N}_0$ for all $j = 1, \cdots, n$, by $[\mathfrak{b}]_n, k$ the $n+1$-tuple $i_1, \cdots, i_n, k$ and by $[\mathfrak{b}]_0$ the empty tuple.

Proposition 3.5. *If the hypothesis of Theorem 3.3 holds, then, for each $m \geq 0$ and each m-tuple $[\mathfrak{b}]_m$, there is a strictly increasing sequence of real numbers*

$$\left(x_{[\mathfrak{b}]_m,k}\right)_{k\geq 0} = \left(x_{i_1,\ldots,i_m,k}\right)_{k\geq 0} \text{ in } \mathfrak{D}_t$$

such that:

(P1) (Convergence) $x_k \to \mathfrak{p}_t^-$, $x_{[0]_k} \to s_{j^\star}$ *and* $x_{[\mathfrak{b}]_m,k} \to x_{[\mathfrak{b}]_m}$ *as* $k \to +\infty$.

(P2) (Contraction) *There is* $\tau \in (0,1)$ *such that*

$$diam\left((x_{[b]_m,k})_k\right) = x_{[b]_m} - x_{[b]_m,0} \leq \tau^m.$$

(P3) (Overlapping) *For every* $h \geq 1$, $x_{[\mathfrak{b}]_m,h,0} < x_{[\mathfrak{b}]_m,(h-1)}$.

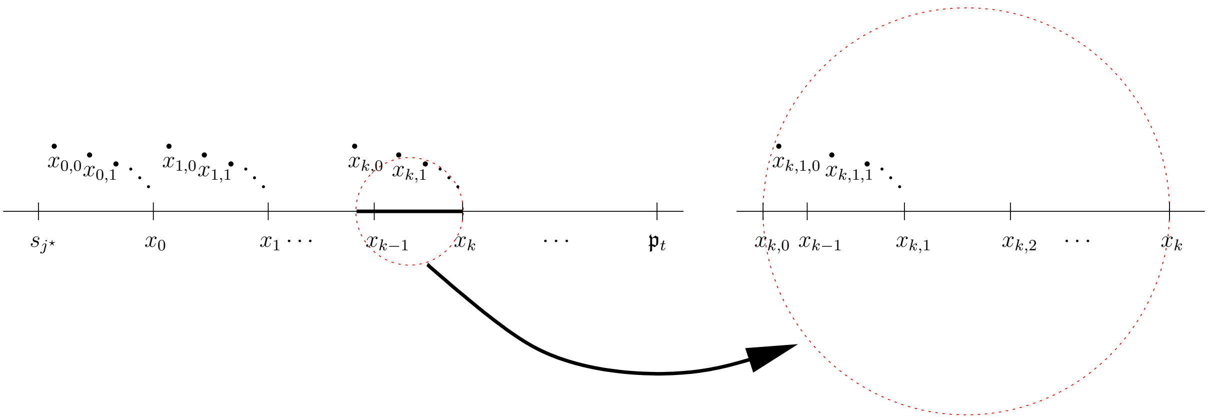

Figure 3.5: The multisequences in Proposition 3.5

The proof is divided into several steps and it is presented below. As an immediate consequence of the Proposition above we get the following corollary.

Corollary 3.6. ([D95b, Lemma 4.1]) *If the hypothesis of Theorem 3.3 holds, then the set*

$$S = \bigcup_{m\geq 1} S_m, \quad \textit{where } S_m := \{x_{[\mathfrak{b}]_m} = x_{i_1,i_2,\ldots,i_m} : (i_1, i_2, \ldots, i_m) \in \mathbb{N}_0^m\},$$

is dense in the interval $[s_{j^\star}, \mathfrak{p}_t] \supset \mathfrak{D}_t^0$.

In the proof of Proposition 3.5, we construct the multisequences $x_{[\mathfrak{b}]_m}$, in order to apply the Corollary 3.6.

Proof of Proposition 3.5. Let $j^\star > j_1$ be such that the map $\Gamma_t^{(0,j^\star)}$ has an attractive fixed point $s_{j^\star} \in \left(g_{0,t}^{-2}(\mathfrak{p}_t), g_{0,t}^{-1}(\mathfrak{p}_t)\right)$. As announced above, since $g_{0,t}$ and $g_{[\omega_0\cdots\omega_{k_0}],t}$ preserve the orientation one gets

$$\Gamma_t^{(0,j^\star+k)}(\mathfrak{D}_t) \subseteq \mathfrak{D}_t, \quad \text{for all } k \in \mathbb{N}.$$

We observe that there exists $m_0 \in \mathbb{N}$ such that

$$\left(\Gamma_t^{(0,j^\star)}\right)^{m_0}(\mathfrak{p}_t) \in [s_{j^\star}, g_{0,t}^{-1}(\mathfrak{p}_t)) \subseteq \mathfrak{D}_t^1.$$

By definition of $s_{j^\star}$, we have

$$\tau := \left(\Gamma_t^{(0,j^\star)}\right)'(s_{j^\star}) < 1,$$

and consequently

$$\max\left\{\left(\Gamma_t^{(0,j^\star+k)}\right)'(x), \left((\Gamma_t^{(0,j^\star)})^{m_0}\right)'(x)\right\} < \tau,$$

where $k \in \mathbb{N}$ and $x \in [s_{j^\star}, \mathfrak{p}_t]$.

We define the zero generation sequence by

$$\begin{cases} x_0 = \left(\Gamma_t^{(0,j^\star)}\right)^{m_0}(\mathfrak{p}_t) \\ x_k = \Gamma_t^{(0,j^\star+k)}(\mathfrak{p}_t), \forall k \geq 1 \end{cases}.$$

Then, we have that $x_0 \notin \mathfrak{D}_t^0$ and the sequence $(x_k)_{k\geq 0}$ is a increasing sequence converging to $\mathfrak{p}_t$.

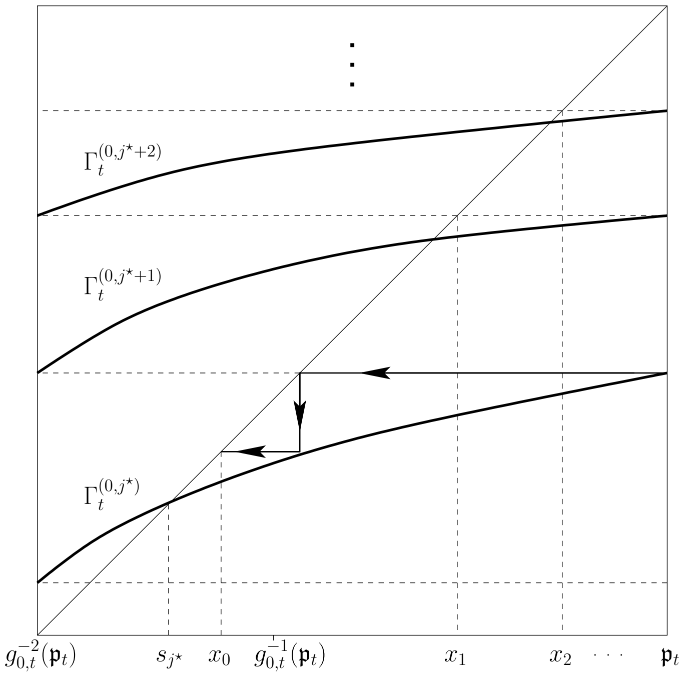

Figure 3.6: The zero generation sequence

The sequence of the first generation is defined by

$$\begin{cases} x_{0,k} = \left(\Gamma_t^{(0,j^\star)}\right)^{m_0}(x_k), \ \forall k \geq 0 \\ x_{i,k} = \Gamma_t^{(0,j^\star+i)}(x_k), \ \forall i \geq 1, \forall k \geq 0 \end{cases}.$$

Since $\lim_{k\to+\infty} x_k = \mathfrak{p}_t$, by continuity, we have that

$$\lim_{k\to\infty} x_{i,k} = \lim_{k\to+\infty} \Gamma_t^{(0,j^\star+i)}(x_k) = \Gamma_t^{(0,j^\star+i)}(\mathfrak{p}_t) = x_i,$$

for each $i \neq 0$. The case $i = 0$ follows analogously. Clearly, $x_{i,k} \geq s_{j^\star}$, for $i, k \in \mathbb{N}_0$. Moreover

$$\begin{aligned} \text{diam}(x_{i,k})_k &= x_i - x_{i,0} \\ &= \Gamma_t^{(0,j^\star+i)}(\mathfrak{p}_t) - \Gamma_t^{(0,j^\star+i)}(x_0) \\ &= (\Gamma_t^{0,j^\star+i})'(\zeta))(\mathfrak{p}_t - x_0), \text{ with } \zeta \in [s_{j^\star}, \mathfrak{p}_t] \\ &\leq \tau\text{diam}(x_k)_k < \tau < 1. \end{aligned}$$

Thus, the first generation satisfies **(P1)** and **(P2)**.

Assume now, inductively, that , for every $l \leq n$ and every ℓ-tuple $[\mathfrak{b}]_\ell$, the sequences $(x_{[\mathfrak{b}]_\ell,k})$ of ℓ-th generation satisfying **(P1)** e **(P2)**:

$\mathbf{(P1)}_\ell$ $(x_{[\mathfrak{b}]_\ell,k})_k$ é is a strictly increasing sequence which converges to $x_{[\mathfrak{b}]_l}$;

$\mathbf{(P2)}_\ell$ $\mathrm{diam}(x_{[\mathfrak{b}]_l,k}) \leq \tau^l$.

We claim that the sequences $(x_{i,[\mathfrak{b}],k})_k$ of generation $n+1$ defined by

$$\begin{cases} x_{0,[\mathfrak{b}]_n,k} = \left(\Gamma_t^{(0,j^\star)}\right)^{m_0}(x_{[\mathfrak{b}]_n,k}) \\ x_{i,[\mathfrak{b}]_n,k} = \Gamma_t^{(0,j^\star+i)}(x_{[\mathfrak{b}]_n,k}), \forall i \geq 1 \end{cases}$$

satisfy $\mathbf{(P1)_{n+1}}$ and $\mathbf{(P2)_{n+1}}$.

Note that, by $\mathbf{(P1)_n}$, $x_{[\mathfrak{b}]_n,k} \to x_{[\mathfrak{b}]_n}$. Thus, for each $i \in \mathbb{N}$ (the case $j = 0$ follows analogously),

$$x_{i,[\mathfrak{b}]_n,k} = \Gamma_t^{(0,j^\star+i)}(x_{[\mathfrak{b}]_n,k}) \to \Gamma_t^{(0,j^\star+i)}(x_{[\mathfrak{b}]_n}) = x_{i,[\mathfrak{b}]_n}.$$

Since $\Gamma_t^{(0,j^\star+i)}$ preserves the orientation and $(x_{[\mathfrak{b}]_n,k})$ is strictly increasing, we conclude that the sequence $(x_{i,[\mathfrak{b}]_n,k})_k$ is also increasing, and so

$\mathbf{(P1)_{n+1}}$ $(x_{i,[\mathfrak{b}]_n,k})_k$ is a strictly increasing sequence which converges to $x_{i,[\mathfrak{b}]_n}$

is proved.

By the definition of $x_{i,[\mathfrak{b}]_n,k}$ on has

$$\begin{aligned} \mathrm{diam}((x_{i,[\mathfrak{b}]_n,k})) &= \mathrm{diam}(\Gamma_t^{(0,j^\star+i)}(x_{[\mathfrak{b}]_n,k})_k) \\ &\leq \tau \mathrm{diam}((x_{[\mathfrak{b}]_n,k})_k) \leq \tau \times \tau^n = \tau^{n+1}, \end{aligned}$$

which implies

$\mathbf{(P2)}_{n+1}$ $\mathrm{diam}((x_{i,[\mathfrak{b}]_n,k})_k) \leq \tau^{n+1}$.

Thus the sequence $(x_{i,[\mathfrak{b}]_n,k})_k$ satisfies **(P1)** and **(P2)** in Proposition 3.5. Next we verify, by induction, that these sequences also satisfy **(P3)** (overlapping condition).

For each $k \geq 1$, one has

$$\begin{aligned} x_{k,0} = \Gamma_t^{(0,j^\star+k)}(x_0) &< \Gamma_t^{(0,j^\star+k)}(g_{0,t}^{-1}(\mathfrak{p}_t)) \\ &= \Gamma_t^{(0,j^\star+k-1)}(\mathfrak{p}_t) = x_{k-1}, \end{aligned}$$

thus the sequence of generation zero satisfies **(P3)**.

To verify **(P3)** for the generation $n+1$, note that by induction

$$x_{[\mathfrak{b}]_n,k,0} < x_{[\mathfrak{b}]_n,(k-1)}, \text{ for } k \geq 1.$$

Thus, since $\Gamma_t^{(0,j^\star+i)}$ is increasing, for all $i \geq 1$, one has

$$x_{i,[\mathfrak{b}]_n,k,0} = \Gamma_t^{(0,j^\star+i)}\left(x_{[\mathfrak{b}]_n,k,0}\right) < \Gamma_t^{(0,j^\star+i)}(x_{[\mathfrak{b}]_n,k-1}) = x_{i,[\mathfrak{b}]_n,k-1}.$$

Therefore, the construction of the sequences verifies Proposition 3.5 and the set

$$S = \bigcup_{m\geq 1} S_m, \quad \text{where } S_m = \{x_{[\mathfrak{b}]_m} = x_{i_1,i_2,\dots,i_m} : (i_1, i_2, \dots, i_m) \in \mathbb{N}_0^m\}$$

is dense in $\left[\lim_{n\to\infty} x_{[0]_k}, \mathfrak{p}_t\right] = [s_{j^\star}, \mathfrak{p}_t]$. □

A straightforward consequence of this proposition is the next corollary.

Corollary 3.7. *For every interval $I = [i_1, i_2] \subset [q_t, p_t)$, there are $x \in (i_1, i_2]$ and a finite sequence $(\vartheta_1, \dots, \vartheta_r)$, $\vartheta_i \in \{0, \cdots, n-1\}$ such that*

$$x = g_{[\vartheta_1\cdots\vartheta_r],t}(\mathfrak{p}_t) = g_{[\omega_0\cdots\omega_{k_0}\vartheta_1\cdots\vartheta_r],t}(p_t).$$

Thus, for every sequence $(\eta_i)_{i\in\mathbb{N}_0}$, with $\eta_i \in \{0, \dots, n-1\}$,

$$X = (0^{-\mathbb{N}}\omega_0\cdots\omega_{k_0}\vartheta_1\cdots\vartheta_r.\eta_0\eta_1\cdots, x) \in W^u(P_t, G_t).$$

Proof. Let $I = [i_1, i_2]$ a subinterval of (q_t, p_t). Then two possibilities can occur: either there is $m = m(I) \in \mathbb{N}_0$ such that $g_{0,t}^{-m}\left((i_1, i_2]\right) \cap \mathfrak{D}_t^0 \neq \emptyset$ or $I \subset [q_t, g_{0,t}^{-1}(\mathfrak{p}_t)]$.

Case 1. There is $m = m(I) \in \mathbb{N}_0$ such that $g_{0,t}^{-m}\left((i_1, i_2]\right) \cap \mathfrak{D}_t^0 \neq \emptyset$.

Since the set S is dense in $\mathfrak{D}_t^0 = (g_{0,t}^{-1}(\mathfrak{p}_t), \mathfrak{p}_t]$, there are $a \in g_{0,t}^{-m}((i_1, i_2]) \cap S$ and a sequence $(\upsilon_1, \dots, \upsilon_l) \in \{0, \dots, n-1\}^l$, with $l \in \mathbb{N}$, such that $a = g_{[\upsilon_1\cdots\upsilon_l],t}(\mathfrak{p}_t)$. Set $x = g_{0,t}^m(a) \in (i_1, i_2]$, therefore $x = g_{[\upsilon_1\cdots\upsilon_l 0^m],t}(\mathfrak{p}_t)$.

Once $\mathfrak{p}_t = g_{[\omega_0\cdots\omega_{k_0}],t}(p_t)$, it follows that, for every $(\eta_i)_{i\in\mathbb{N}_0}$ with $\eta_i \in \{0, \dots, n-1\}$,

$$X = (0^{-\mathbb{N}}\omega_0\cdots\omega_{k_0}\upsilon_1\cdots\upsilon_l 0^{-m}.\eta_0\eta_1\cdots, x) \in W^u(P_t, G_t).$$

Case 2. $I \subset [q_t, g_{0,t}^{-1}(\mathfrak{p}_t)]$

In this case we have $g_{[\omega_0\cdots\omega_{k_0}.],t}(I) \subset (\mathfrak{p}_t, p_t)$ and consequently, there is $m \in \mathbb{N}$ such that

$$g_{0,t}^{-m}\Big(g_{[\omega_0\cdots\omega_{k_0}.],t}\left((i_1,i_2]\right)\Big) \cap \mathfrak{D}_t^0 \neq \emptyset$$

and the result follows in similar way as in the previous case. □

Finally we are in position to prove the inclusion $H_{\mathcal{V}}(Q_t, G_t) \subset H_{\mathcal{V}}(P_t, G_t)$.

Proof of Theorem 3.3. It is enough to prove that every transverse homoclinic point $X \in W^s(Q_t, G_t) \pitchfork W^u(Q_t, G_t)$ is accumulated by points in $H_{\mathcal{V}}(P_t, G_t)$. Replacing X by some iterate of it, we can assume that

$$X = (0^{-\mathbb{N}}\upsilon_{-r}\cdots\upsilon_{-1}.0^{\mathbb{N}}, q_t).$$

Since $X \in W^u(Q_t, G_t)$, there is $m_0 \in \mathbb{N}$ such that $g_{[\upsilon_{-r}\cdots\upsilon_{-1}.],t}([q_t, q_t + 1/m]) \subset [q_t, p_t)$, for every $m \geq m_0$.

Denote by $I_m := g_{[0^{-m}\upsilon_{-r}\cdots\upsilon_{-1}.],t}([q_t, q_t + 1/m])$, with $m \geq m_0$. By Corollary 3.7, there are $x_m \in I_m$, $x_m \neq q_t$, and a sequence $(\vartheta_1, \ldots, \vartheta_l)$, with $\vartheta_i \in \{0, \ldots, n-1\}$ for every $i = 1, \ldots, l$, such that $x_m = g_{[\vartheta_1\cdots\vartheta_l],t}(\mathfrak{p}_t)$. Thus,

$$X_m = (0^{-\mathbb{N}}\omega_0\cdots\omega_{k_0}\vartheta_1\cdots\vartheta_l 0^{-m}\upsilon_{-r}\cdots\upsilon_{-1}.0^{\mathbb{N}}, \overline{x}_m) \in W^s(P_t, G_t) \pitchfork W^u(P_t, G_t),$$

with $\overline{x}_m = g_{[0^m\upsilon_{-r}\cdots\upsilon_{-1}],t}(x_m) \in [q_t, q_t + 1/m]$. Hence X_m (thus X) belongs to $H_{\mathcal{V}}(P_t, G_t)$, ending the proof of the theorem. □

In what follows we present the similar result to heterodimensional cycles. Thus, we consider the one-parameter family of diffeomorphisms $(f_t)_{t\geq 0}$, with t small. In this setting we recall that $X_t = (-1, t, 0)$ is a transverse homoclinic point of $P = (0, 1/2, 0)$ and the goal is to construct a set $\mathcal{H}_t$ of homoclinic points of P with the form $(x, y, 0)$ such that the set $\{y : (x, y, 0) \in \mathcal{H}_t\}$ is dense in a fundamental domain of F and it is contained in the $\mathfrak{F}_t$-orbit of t.

Theorem 3.8. *Consider $t > 0$ and assume that there is $j^\star$ such that the map $\Phi_t^{0,j^\star}$ has a attractive fixed point $\tilde{s_{j^\star}} \in (F^{-2}(t), F^{-1}(t))$. Then $H(Q, f_t) \subseteq H(P, f_t)$.*

It is convenient to note that if the eigenvalue λ is close to 1, the amplitude of the interval $D_P^t = (1/2 - t, F(1/2 - t)]$ is close to 0 and $F_{1,t}(1/2 - \lambda t)$ goes to zero as

$\lambda \to 1^-$. Therefore, to get the hypothesis of theorem holds, it is sufficient to choose λ close to 1.

The next proposition is an adaptation to the context of heterodimensional cycles of the Proposition 3.5 and the proof is done using similar arguments, so it is omitted.

Proposition 3.9. *If the hypothesis of Theorem 3.8 holds, then, for each $m \in \mathbb{N}$ and each m-tuple $[\mathfrak{b}]_m$, there are sequences of homoclinic points P of the form*

$$\left(a_{[\mathfrak{b}]_n,k}, y_{[\mathfrak{b}]_n,k}, 0\right), \ a_{[\mathfrak{b}]_n,k} \in [-1,0], \ y_{[\mathfrak{b}]_n,k} \in (F^{-2}(t), t],$$

such that $(y_{[\mathfrak{b}]_m,k})_{k\geq 0}$ is a strictly increasing sequence, $y_0 \in (F^{-2}(t), F^{-1}(t))$, and

P1) (Convergence) $y_k \to t^-$, $y_{[0]_k} \to \tilde{s_{j^\star}}$, *and* $y_{[\mathfrak{b}]_m,k} \to y_{[\mathfrak{b}]_m}$ *as* $k \to \infty$;

P2) (Contraction) $\operatorname{diam}\left((y_{[\mathfrak{b}]_m,k})_k\right) \to 0$ *as* $m \to +\infty$.

P3) (Overlapping) *For every* $h \geq 1$, $y_{[b]_m,h,0} < y_{[b]_m,(h-1)}$.

Let

$$\tilde{S} := \bigcup_{m\geq 1} \tilde{S_m}, \quad \text{where } \tilde{S_m} := \{y_{[\mathfrak{b}]_m} = y_{i_1,i_2,\ldots,i_m} : (i_1, i_2, \ldots, i_m) \in \mathbb{N}_0^m\}.$$

Then the set $\tilde{S}$ is dense in $[\tilde{s_j^\star}, t] \supset [F^{-1}(t), t]$.

Now the proof of Theorem 3.8 follows as in [D95b, section 5].

3.3 Growth of number of periodic orbits

We say that a diffeomorphism $g \in Diff^r(M)$ is Artin-Mazur if $Per_m(g)$ grows at most exponentially fast, i.e. there is a constant $C > 0$ such that

$$Per_m(g) \leq exp(Cm), \quad \text{for all } m \in \mathbb{N},$$

where $Per_m(g)$ denotes the number of isolated periodic points of period m of g. In this section we prove that the number of periodic points of period n of $G_t|_{\Lambda_t}$ grows at most exponentially fast, that is, $G_t|_{\Lambda_t}$ is Artin-Mazur. We also get a similar result for the one-parameter family $(f_t)_{t\geq 0}$.

Recall the definition of the maps $\Gamma_t^{(u,s)}$ in (1.3.16), with $(u,s) \in \mathbb{N} \times \mathbb{N}$, and note that $u \geq u_0$ and $s \geq h$ where h and u_0 are defined by

$$g_{[0^h \omega_0 \cdots \omega_{k_0}],t}(d_t) = q_t \text{ and } g_{[\omega_0 \cdots \omega_{k_0} 0^{u_0}],t}(p_t) \in D_t,$$

respectively. We will denote v and w by $v := u - u_0$ and $w := s - h$.

Lemma 3.10. *For all $v,\ w \geq 0$ the map $\Gamma_t^{(v,w)} : D_t^{(v,w)} \to D_t$ has at most two fixed points.*

Proof. First note that the maps $\Gamma_t^{(v,w)}$ are compositions of $g_{0,t}$ and $g_{[\alpha_0,\ldots,\alpha_r],t}$, which preserve the orientation, respectively, in $\mathbb{K}$ and in a neighborhood of p_t, therefore the maps $\Gamma_t^{\mathfrak{b}}$ are also order preserving. Moreover, as $g'_{0,t}$ is a decreasing map on $[0,1/2]$ and $\omega_0 \cdots \omega_{k_0} = 0^k \alpha_0 \cdots \alpha_r 0^k$, for k large, the map $\left(\Gamma_t^{(v,w)}\right)'$ is also decreasing and consequently it has at most two fixed points on $D_t^{(v,w)}$. □

We denote by $\mathcal{H}$ the set $\mathbb{N}_0 \times \mathbb{N}_0$ and, for each point $(v,w) \in \mathcal{H}$, we associate the map $\Gamma_t^{(v+u_0,w+h)} : D_t^{(v+u_0,w+h)} \to D_t$.

For $i,\ \alpha \in \mathbb{N}$, let

$$\mathcal{H}(i,\alpha) := \{[(v_1,w_1),\ldots,(v_i,w_i)] \in \mathcal{H}^i,\ v_1 + w_1 + \ldots + v_i + w_i = \alpha\}$$

and $P(i,\alpha)$ the number of elements of $\mathcal{H}(i,\alpha)$, i.e., $P(i,\alpha) := \#\mathcal{H}(i,\alpha)$.

Note that we can write $\mathcal{H}(i,\alpha)$ as the disjoint union

$$\mathcal{H}(i,\alpha) = \bigcup_{j=0}^{\alpha} \mathcal{H}(1,j) \times \mathcal{H}(i-1,\alpha-j),$$

so the cardinality of $\mathcal{H}(i,\alpha)$ is equal to

$$P(i,\alpha) = P(1,0) \cdot P(i-1,\alpha) + P(1,1) \cdot P(i-1,\alpha-1) + \cdots + P(1,\alpha) \cdot P(i-1,0),$$

thus

$$P(i,\alpha) = 1P(i-1,\alpha) + 2P(i-1,\alpha) + \cdots + \alpha P(i-1,1) + \alpha + 1, \qquad (3.3.1)$$

since $P(1,\beta) = \beta + 1$ for all $\beta \geq 0$ and $P(i-1,0) = 1$.

Before to prove that $G_t|_{\Lambda_t}$ is Artin-Mazur, we need to introduce the following definition and the technical Lemma 3.11 below.

Definition 3.1. *Given $i \in \mathbb{N}$, we say that i verifies the* property $\mathcal{P}$ *if*

$$P(1, \alpha_1 + 1) + \cdots + P(i, \alpha_i + 1) < 2\Big(P(1, \alpha_1) + \cdots + P(i, \alpha_i)\Big),$$

for every sequence $(\alpha_j)_{j=1}^i \in \mathbb{N}$ satisfying $\alpha_{j+1} - \alpha_j > 2$, for each $j = 1, \ldots, i-1$.

Lemma 3.11. *The property $\mathcal{P}$ holds for all $n \in \mathbb{N}$.*

By technical reasons the proof of the lemma is postponed at the end of the chapter.

We are now in position to prove that $G_t|_{\Lambda_t}$ is Artin-Mazur.

Proposition 3.12. *For all $t > 0$ small enough one has*

$$Per_m(G_t|_{\Lambda_t}) \leq 2^{m-(h+u_0)+1}, \quad \textit{for all } m \geq h + u_0.$$

In particular, $G_t|_{\Lambda_t}$ is an Artin-Mazur diffeomorphism.

Proof. Let $m \in \mathbb{N}$, such that $m \geq n_0$, where $n_0 := h + u_0$. If $m = n_0$, from Lemma 3.10, one has $Per_{m_0}(G_t|_{\Lambda_t}) \leq 2$. Otherwise, the number m can be written in the following way. Let $k = [m/n_0]$ and for each $j = 0, \ldots, k$ we write

$$m = jn_0 + r_j, \;\; r_j \geq 0,$$

where

$$0 \leq r_k < n_0.$$

This gives all the possible itineraries of a periodic point of period m.

For each $i \in \{1, \cdots, k\}$ and to each element $[(v_1, w_1), \ldots, (v_i, w_i)]$ of $\mathcal{H}(i, r_i)$ we can associate the chain of pairs $\mathfrak{b}_i = (v_1 + u_0, w_1 + h) \cdots (v_i + u_0, w_i + h)$ and the map

$$\Gamma_t^{\mathfrak{b}_i} : D_t^{\mathfrak{b}_i} \to D_t, \quad x \mapsto g_{[\theta(\mathfrak{b}_i), t]}(x).$$

We observe that there exist chains $\mathfrak{b}_i$ such that $D_t^{\mathfrak{b}_i} = \emptyset$.

Defining

$$I_{G_t|_{\Lambda_t}}(m) := P(1, r_1) + P(2, r_2) + \cdots + P(k, r_k),$$

then $I_{G_t|_{\Lambda_t}}(m)$ corresponds to all possible itineraries that provide a periodic point of period m. Note that some of these itineraries have no periodic points. Using Lemma 3.10, one gets

$$Per_m(G_t|_{\Lambda_t}) \leq 2 \cdot I_{G_t|_{\Lambda_t}}(m).$$

Now the result follows immediately if we prove the following:

Claim. *For all $t > 0$ small enough and $m \geq 0$ it holds $I_{G_t|_{\Lambda_t}}(m) \leq 2^{m-n_0}$ for all $m \geq n_0$*

Proof. We argue inductively on m.

If $m = n_0$, then $I_{G_t|_{\Lambda_t}}(m) = P(1,0) = 1$, corresponding to the map $\Gamma_t^{(u_0,h)}$.

Assume that the claim is true for m, that is,

$$I_{G_t|_{\Lambda_t}}(m) = P(1,r_1) + P(2,r_2) + \cdots + P(k,r_k) \leq 2^{m-n_0}.$$

We now get the estimate for $m+1$. Two possibilities could happen: $0 \leq r_k < n_0 - 1$ or $r_k = n_0 - 1$.

Case 1: $r_k < n_0 - 1$

If $r_k < n_0 - 1$ then $k(m) = k(m+1)$ and the possibilities for "the splitting for $m+1$ " are

$$m + 1 = jn_0 + r_j + 1, \ j = 1, \dots, k,$$

that is we have the same combinatory as the one we had for m.

Since $r_{j+1} - r_j = n_0 > 2$, for each $j = 1, \dots, k-1$, by Lemma 3.11,

$$\begin{aligned} I_{G_t|_{\Lambda_t}}(m+1) &= P(1, r_1 + 1) + \cdots + P(k, r_k + 1) \\ &< 2\left(P(1,r_1) + \cdots + P(k,r_k)\right), \end{aligned} \tag{3.3.2}$$

and, by induction, we are done, that is, $I_{G_t|_{\Lambda_t}}(m+1) \leq 22^{m-n_0} = 2^{(m+1)-n_0}$.

Case 2: $r_k = n_0 - 1$

In this case, the possible splitting for $m+1$ is

$$\begin{cases} m + 1 = jn_0 + r_j + 1, \ j = 1, \ \dots, k \\ m + 1 = (k+1)n_0, \end{cases},$$

thus

$$\begin{aligned} I_{G_t|_{\Lambda_t}}(m+1) &= P(1,r_1+1)+\cdots+P(k,r_k+1)+P(k+1,0) \\ &= P(1,r_1+1)+\cdots+P(k,r_k+1)+1 \\ &\leq 2\left(P(1,r_1)+\cdots+P(k,r_k)\right), \end{aligned} \tag{3.3.3}$$

where the last inequality follows from Lemma 3.11.

By induction hypothesis, from (3.3.2) and (3.3.3) we conclude that

$$I_{G_t|_{\Lambda_t}}(m+1) \leq 2I_{G_t|_{\Lambda_t}}(m) \leq 2\cdot 2^{m-n_0} = 2^{m+1-n_0},$$

ending the proof of the claim. □

Finally, let us prove Lemma 3.11.

Proof of Lemma 3.11. We shall show by complete induction that the property $\mathcal{P}$ holds for all $n \in \mathbb{N}$.

As

$$P(1,\alpha+1) = \alpha+2 < 2(\alpha+1) = 2P(1,\alpha),\ \forall \alpha\in\mathbb{N}.$$

the property $\mathcal{P}$ is trivial verified for $n=1$.

Now, fix $l \in \mathbb{N}$ and assume that the property holds for all m less than l. We shall show that it holds for l. For that, taking a sequence $(\alpha_j)_{j=1}^{l}$ verifying

$$\alpha_{j+1}-\alpha_j > 2, \quad \text{for each } j=1,\ldots,l-1,$$

the goal is to prove that

$$P(1,\alpha_1+1)+P(2,\alpha_2+1)+\cdots+P(l,\alpha_l+1) < 2\Big(P(1,\alpha_1)+P(2,\alpha_2)+\cdots+P(l,\alpha_l)\Big).$$

Applying (3.3.1) to $P(2,\alpha_2),\cdots,P(l,\alpha_l)$, we have that

$$\begin{aligned} &P(1,\alpha_1+1)+P(2,\alpha_2+1)+\cdots+P(l,\alpha_l+1) = \\ = \ &P(1,\alpha_1+1)+\Big(P(1,\alpha_2+1)+2P(1,\alpha_2)+\cdots+(\alpha_2+2)P(1,0)\Big) \\ &+\cdots+\Big(P(l-1,\alpha_l+1)+2P(l-1,\alpha_l)+\cdots+(\alpha_l+2)P(l-1,0)\Big) \end{aligned}$$

Reorganizing the terms of the sum and applying the hypothesis, we have

$$\begin{aligned}
& P(1,\alpha_1+1)+\Big(P(1,\alpha_2+1)+P(2,\alpha_3+1)+\cdots+P(l-1,\alpha_l+1)\Big)+\\
& +2\Big(P(1,\alpha_2)+P(2,\alpha_3)+\cdots+P(l-1,\alpha_l)\Big)+\cdots+\\
& +\alpha_l\Big(P(1,\alpha_2-\alpha_l+2)+P(2,\alpha_3-\alpha_l+2)+\cdots+P(l-1,2)\Big)\\
& +(\alpha_l+1)\Big(P(1,\alpha_2-\alpha_l+1)+P(2,\alpha_3-\alpha_l+1)+\cdots+P(l-1,1)\Big)+\\
& +(\alpha_l+2)\Big(P(1,\alpha_2-\alpha_l)+P(2,\alpha_3-\alpha_l)+\cdots+P(l-2,1)+1\Big)+\\
& +\cdots+\alpha_2P(1,2)+(\alpha_2+1)P(1,1)+\alpha_2+2\\
< \; & P(1,\alpha_1+1)+2\Big(P(1,\alpha_2)+P(2,\alpha_3)+\cdots+P(l-1,\alpha_l)\Big)+\\
& +2^2\Big(P(1,\alpha_2-1)+P(2,\alpha_3-1)+\cdots+P(l-1,\alpha_l-1)\Big)+\cdots+\\
& +2\alpha_l\Big(P(1,\alpha_2-\alpha_l+1)+P(2,\alpha_3-\alpha_l+1)+\cdots+P(l-1,1)\Big)\\
& +2(\alpha_l+1)\Big(P(1,\alpha_2-\alpha_l)+P(2,\alpha_3-\alpha_l)+\cdots+P(l-1,0)\Big)+\\
& +(\alpha_l+2)\Big(P(1,\alpha_2-\alpha_l-1)+P(2,\alpha_3-\alpha_l-1)+\cdots+P(l-2,0)\Big)+\\
& +\cdots+2(\alpha_2)P(1,1)+2(\alpha_2+1)P(1,0)+\alpha_2+2.
\end{aligned}$$

Now reorganizing again the terms of the sum we obtain

$$\begin{aligned}
& P(1,\alpha_1+1)+P(2,\alpha_2+1)+\cdots+P(l,\alpha_l+1)\\
< \; & P(1,\alpha_1+1)+\alpha_2+2+2\Big(P(1,\alpha_2)+2P(1,\alpha_2-1)+\cdots+\\
& +(\alpha_2+1)P(1,0)\Big)+\cdots+2\Big(P(l-1,\alpha_l)+2P(l-1,\alpha_l-1)+\cdots+\\
& +(\alpha_l+1)P(l-1,0)\Big).
\end{aligned}$$

Recalling that

$$P(i,\alpha_i)=1P(i-1,\alpha_i)+2P(i-1,\alpha_i-1)+\cdots(\alpha_i+1)P(i-1,0)$$

and $P(1,\alpha_1+1)=\alpha_1+2$, we obtain

$$\begin{aligned}
& P(1,\alpha_1+1)+P(2,\alpha_2+1)+\cdots+P(l,\alpha_l+1)\\
< \; & P(1,\alpha_1+1)+\alpha_2+2+2P(2,\alpha_2)+2P(3,\alpha_3)+\cdots+2P(l,\alpha_l)\\
= \; & P(1,\alpha_1+1)+\alpha_2+2+2\Big(P(2,\alpha_2)+\cdots+P(l,\alpha_l)\Big)\\
= \; & \alpha_1+2+\alpha_2+2+2\Big(P(2,\alpha_2)+\cdots+P(l,\alpha_l)\Big)\\
< \; & 2\,(\alpha_1+1)+2\Big(P(2,\alpha_2)+\cdots+P(l,\alpha_l)\Big),
\end{aligned}$$

where the last equality follows from $\alpha_1 - 2 > \alpha_2$, thus

$$\begin{aligned} & P(1,\alpha_1+1)+P(2,\alpha_2+1)+\cdots+P(l,\alpha_l+1) \\ < \; & 2P(1,\alpha_1)+2\Big(P(2,\alpha_2)+\cdots+P(l,\alpha_l)\Big) \\ = \; & 2\Big(P(1,\alpha_1)+P(2,\alpha_2)+\cdots+P(l,\alpha_l)\Big), \end{aligned}$$

Consequently the property $\mathcal{P}$ holds for l as required, ending the proof of the lemma. □

For heterodimensional cycles, we observe that F is strictly increasing and F' is strictly decreasing, so for each $(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0$ the map $\Phi_t^{m,n}$ has at most two fixed points. Consequently we have the following result whose proof we omit since is identical to the proof of Proposition 3.12. Recall the definition of k_0 and note that N_t is such that $F^{N_t}(D_t^Q) = D_t^P$.

Proposition 3.13. *For all $t > 0$ small enough one has*

$$Per_m(f_t|_{\Lambda_t}) \leq 2^{m-(N_t+k_0)+1}, \; \forall m \geq N + k_0.$$

In particular $f_t|_{\Lambda_t}$ is an Artin-Mazur diffeomorphism.

Chapter 4

Non-hyperbolic ergodic measures

In this chapter, our focus is the construction of non-hyperbolic ergodic measures for the one-parameter family of skew-product maps $G_t : \Sigma_n \times \mathbb{K} \to \Sigma_n \times \mathbb{K}$, with $t \geq 0$ small, introduced in Section 1.3.

Assuming the *Non-hyperbolicity hypothesis* stated below, which in particular implies the expansiveness of the system of iterated functions $\mathfrak{G}_t$ and the inclusion $H_{\mathcal{V}}(P_t, G_t) \subset H_{\mathcal{V}}(Q_t, G_t)$, we show that G_t admits an invariant ergodic measure with one of the Lyapunov exponents equal to zero, called a "non-hyperbolic" ergodic measure. This is the main result of this chapter. To prove it we show that there are positive numbers, δ_t, l_t, and L_t, such that, for every subinterval J of $D_t = (d_t, g_{0,t}(d_t)]$ such that $|J| < \delta_t$, there is a chain $\mathfrak{b}(J)$ such that

$$|J| < l_t|J| < |(\Gamma_t^{\mathfrak{b}(J)})(J)| < L_t|J|, \quad 1 < l_t < L_t.$$

This allowed us to construct a sequence of periodic orbits with increasing periods that verify the assumptions of a result proved in [DG09] (see Proposition 4.3 below) where sufficient conditions for the existence of non-hyperbolic ergodic measures are stated.

4.1 Notation and Definitions

A collection $\mathfrak{B}$ of subsets of a compact manifold M is called a σ-algebra over M if the following conditions hold:

- $M \in \mathfrak{B}$
- if $B \in \mathfrak{B}$, then $M \setminus B \in \mathfrak{B}$
- if $B_n \in \mathfrak{B}$ for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} B_n \in \mathfrak{B}$.

The pair $(M, \mathfrak{B})$ is called *measurable space* and the members of $\mathfrak{B}$ are called *measurable sets*. A *finite measure* on $(M, \mathfrak{B})$ is a function $\nu : \mathfrak{B} \to [0, +\infty)$ satisfying $\nu(\emptyset) = 0$ and for all countable collections $\{B_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in $\mathfrak{B}$,

$$\nu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \nu(B_n).$$

The measure ν is *non-atomic* if, for any measurable set with $\nu(A) > 0$, there exist a measurable subset C of A such that $\nu(A) > \nu(C) > 0$. Naturally, this implies that no singular set has positive measure.

A measurable set A in M is said to be of *full measure in M* if $\nu(M \setminus A) = 0$. The *support of the measure* ν on M, $\mathrm{supp}(\nu)$, is the smallest closed set with full measure, that is,

$$\mathrm{supp}(\nu) = \bigcap \{C : C \text{ is closed, and } \nu(M \setminus C) = 0\}.$$

A map $f : M \to M$ is called a *measurable map* if

$$f^{-1}(A) \in \mathfrak{B}, \quad \text{for all } A \in \mathfrak{B}.$$

Let $f : M \to M$ be a measurable map. A measure ν is *f-invariant* provided $\nu(f^{-1}(A)) = \nu(A)$ for all measurable sets A. Moreover, f is called *ergodic* for an f-invariant measure ν if and only if, for each A measurable such that $f^{-1}(A) = A$ and $\nu(A) > 0$, we have $\nu(M \setminus A) = 0$. In this situation, we say that ν is an ergodic measure of f, that is, for an ergodic map, all invariant measurable sets either have zero measure or full measure.

A measure ν on M is called a *Borel measure* if it is a measure defined on the σ-algebra of Borel sets, that is, the smallest σ-algebra that contains all open subsets of M. An element of the σ-algebra of Borel is called a *Borel subset* of M or a *Borel set*. A measure ν defined on the σ-algebra of Borel is said to be a *Borel probability measure on M* if $\nu(M) = 1$.

We now present the Multiplicative Ergodic Theorem of Oseledec (see [O68]) and give the definition of Lyapunov exponents.

Definition 4.1. *A point $x \in M$ is a* regular point *for f if there are numbers*

$$\lambda_1(x) > \cdots > \lambda_l(x)$$

and a decomposition

$$T_x M = E_1(x) \oplus \cdots \oplus E_l(x)$$

such that

$$\lim_{k \to \infty} \frac{1}{k} \log(\|Df_x^k v\|) = \lambda_j(x)$$

for $v \in E_j$, $v \neq 0$, and $1 \leq j \leq l$.

These values $\lambda_j(x)$ are called the *Lyapunov exponents of f at x*. Thus, $\lambda_j(x)$ is the exponential growth rate of the vector transported by the linearized equations along the orbit and not depend on the length of the vector. The multiplicative Ergodic Theorem states that for every invariant probability measure ν, the set of regular points Λ is a Borel subset and $\nu(\Lambda) = 1$. If the diffeomorphism f is ergodic for an invariant measure ν, then the $\lambda_j' s$ as well as the dimensions of the corresponding spaces E_j's are constant $\nu-$ almost everywhere and we can speak of the *Lyapunov exponents of the diffeomorphism*. The Lyapunov exponents are constant along an orbit by the definition.

Here we consider $M = \Sigma_n \times \mathbb{K}$, with $\mathbb{K} = \mathbb{S}^1$ or $\mathbb{K} = [-1, 1]$, and a skew-product

$$\begin{array}{rccc} G_t : & \Sigma_n \times \mathbb{K} & \to & \Sigma_n \times \mathbb{K} \\ & (\xi, x) & \mapsto & (\sigma(\xi), g_{\xi_0,t}(x)), \end{array}$$

where, recall, $g_{0,t}, \ \cdots, \ g_{n-1,t}$ are diffeomorphisms in $\mathbb{K}$, Σ_n is the base and $\mathbb{K}$ is the fiber of the product. The *Lyapunov exponent along the fiber at a point* (ξ, x) is given by

$$\lambda_t(\xi, x) := \lim_{m \to \infty} \frac{1}{m} \log |(g_{\xi_m,t} \circ g_{\xi_{m-1},t} \circ \cdots \circ g_{\xi_0,t})'(x)|.$$

whenever this limit exists. Note that if the step skew-product G_t is ergodic for an invariant measure ν, then $\lambda_t(\xi, x)$ is constant ν- almost everywhere and we can speak of the Lyapunov exponent along the fiber with respect to the measure ν. We denote it by $\lambda_t(\nu)$.

Finally, we introduce a further definition.

Definition 4.2. *An ergodic invariant measure of a diffeomorphism is called* non-hyperbolic *if at least one of its Lyapunov exponents is zero.*

4.2 Existence of an invariant non-hyperbolic measure

In this section we state the main result of this chapter and we give an idea of its proof. In the next section we present the proof of this result.

Recall the definition of the maps G_t and $\Gamma_t^{(u,s)} : D_t^{(u,s)} \to D_t$ in Section 1.3 and note that d_t and h are such that

$$g_{[0^h \omega_0 \cdots \omega_{k_0}]}(d_t) = q_t.$$

Recall that $D_t = [d_t, g_{0,t}(d_t)]$.

In this chapter we assume that the following conditions, called *Non-hyperbolicity hypothesis* hold:

Non-hyperbolicity hypothesis: There are $j_0 \in \mathbb{N}$, a chain $\mathfrak{b}^\star = (u_1^\star, h) \cdots (u_{j_0}^\star, h)$, and $l_t > 1$, such that:

(NH1) $g_{0,t}(d_t) \in D_t^{\mathfrak{b}^\star}$ and $\Gamma_t^{\mathfrak{b}^\star}(g_{0,t}(d_t)) < g_{0,t}(d_t)$;

(NH2) $\left(\Gamma_t^{\mathfrak{b}^\star}\right)'(g_{0,t}(d_t)) > l_t$; and

(NH3) $\left(\Gamma_t^{(u_i^\star+1,h)} \circ \Gamma_t^{(u_{i-1}^\star,h)} \circ \cdots \circ \Gamma_t^{(u_1^\star,h)}\right)'(d_t^i) > l_t$, where

$$d_t^i := \left(\Gamma_t^{(u_i^\star,h)} \circ \Gamma_t^{(u_{i-1}^\star,h)} \circ \cdots \circ \Gamma_t^{(u_1^\star,h)}\right)^{-1}(d_t),$$

for each $i = 1, \ldots, j_0$.

The next result implies that if G_t satisfies the Non-hyperbolicity hypothesis, then G_t verifies the condition **(EC)** (see Section 2.2) and from Corollary 2.10, we get $H_{\mathcal{V}}(P_t, G_t) \subseteq H_{\mathcal{V}}(Q_t, G_t)$ where $\mathcal{V}$ is a neighborhood of the cycle (see (1.3.10)).

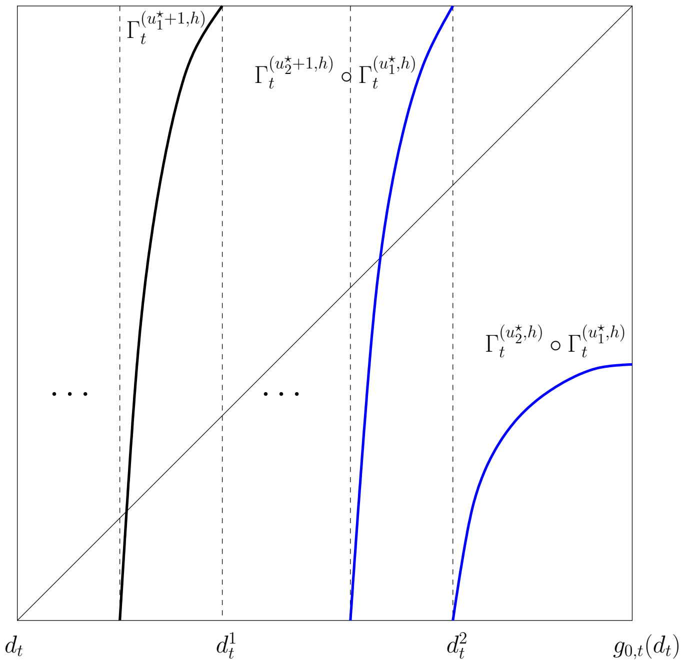

Figure 4.1: The Non-hyperbolicity hypothesis

Lemma 4.1. *Let G_t be a skew-product map that satisfies the Non-hyperbolicity hypothesis. Therefore, for each $x \in (d_t, g_{0,t}(d_t)]$, there is a chain $\mathfrak{b} = \mathfrak{b}(x) = (u_1, h) \cdots (u_i, h)$, $0 < i \le j_0$, such that $\left(\Gamma_t^{\mathfrak{b}}\right)'(x) > l_t$.*

Proof. First observe that $u_1^\star$ is such that $g_{a,t}(d_t) \in \Gamma_t^{(u_1^\star,h)}$ and

$$(d_t, g_{0,t}(d_t)] = \bigcup_{u \ge u_1^\star} D_t^{(u_1,h)} = \bigcup_{u > u_1^\star} D_t^{(u_1,h)} \cup (d_t^1, g_{0,t}(d_t)],$$

where $d_t^1 = \left(\Gamma_t^{(u_1^\star,h)}\right)^{-1}(d_t)$ (see Figure 4.1). Since, by **(NH3)**, $\Gamma_t^{(u_1^\star+1,h)}(d_t^1) > l_t > 1$, then, from the monotonicity of $g_{0,t}$ and $g'_{0,t}$ on $[q_t, p_t]$, we can conclude that

$$\Gamma_t^{(u,h)}(y) > l_t, \quad \text{for all } y \in \bigcup_{u > u_1^\star} D_t^{(u_1,h)} = (d_t, d_t^1].$$

Moreover, we have

$$(d_t^1, g_{0,t}(d_t)] = \bigcup_{u_2 \ge u_2^\star} D_t^{(u_2,h)(u_1^\star,h)} = \bigcup_{u_2 > u_2^\star} D_t^{(u_2,h)(u_1^\star,h)} \cup (d_t^2, g_{0,t}(d_t)],$$

where $d_t^2 = \left(\Gamma_t^{(u_2^\star,h)} \circ \Gamma_t^{(u_1^\star,h)}\right)^{-1}(d_t)$ (see Figure 4.1). From condition **(NH3)**

$$\left(\Gamma_t^{(u_2^\star+1,h)} \circ \Gamma_t^{(u_1^\star,h)}\right)'(d_t^2) > l_t$$

holds, which implies that

$$\left(\Gamma_t^{(u_2,h)} \circ \Gamma_t^{(u_1^\star,h)}\right)'(x) > l_t, \quad \text{for all } x \in \bigcup_{u_2 > u_2^\star} D_t^{(u_2,h)(u_1^\star,h)} = (d_t^1, d_t^2].$$

Proceeding in the same way with $(d_t^2, g_{0,t}(d_t)]$ and so one, we obtain the claim. In fact, given $x \in D_t$, either $x \in D_t^{\mathfrak{b}^\star}$ and we have, from **(NH2)**,

$$\left(\Gamma_t^{\mathfrak{b}^\star}\right)'(x) > \Gamma_t^{\mathfrak{b}^\star}(g_{0,t}(d_t)) > l_t,$$

or there are $0 < i \leq j_0$ and a chain $\mathfrak{b} = \mathfrak{b}(x) = (u_i,h)(u_{i-1}^\star,h)\cdots(u_1^\star,h)$, with $u_i \geq u_i^\star + 1$, such that

$$\left(\Gamma_t^{\mathfrak{b}}\right)'(x) > \left(\Gamma_t^{(u_i^\star+1,h)} \circ \Gamma_t^{(u_{i-1}^\star,h)} \circ \cdots \circ \Gamma_t^{(u_1^\star,h)}\right)'(d_t^i) > l_t,$$

ending the proof of the lemma. □

Let us now see that the map $G_{a,t}$ satisfies the Non-hyperbolicity hypothesis for $a \in (0, \log 2)$ and $t \in (0, t_0(a)]$, and $(a,t) \in (\log 2, \log 4) \times (t_n(a) - \alpha_{a,t_n}, t_n(a) + \alpha_{a,t_n})$, that is, the values considered in Chapter 2. Recall the definition of $G_{a,t}$ in Section 2.1 and note that $g_{0,t} = g_a$ and $D_{a,t} = \left[g_a^k(d_{a,t}), g_a^{k+1}(d_{a,t})\right]$.

For $a \in (0, \log((1+\sqrt{5})/2))$, we have $g_a^{k+1}(c_t) \in D_{a,t}^{(u_1,n-2k)}$ and

$$\left(\Gamma_{a,t}^{(u_1,n-2k)}\right)'(g_a^{k+1}(d_{a,t})) > A > 1,$$

where $u_1 \geq 3$, and, from the monotonicity of g_a and g_a', it follows immediately that $G_{a,t}$ verifies the Non-hyperbolicity hypothesis with $\mathfrak{b}^\star = (u_1, n-2k)$. The skew-product map $G_{a,t}$ also satisfies the Non-hyperbolicity hypothesis for $a \in [\log((1+\sqrt{5})/2), \log 2]$ with $j_0 = 2$ and $\mathfrak{b}^\star = (2, n-2k)(u_2, n-2k)$, where $u_2 \geq 2$. Finally, for $a \in (\log 2, \log 4)$ and $t \in (t_n(a) - \alpha_{a,t_n}, t_n(a) + \alpha_{a,t_n})$, it is easy to see that conditions **(P1')-(P4')** (see Section 2.3) imply the Non-hyperbolicity hypothesis.

Let us now state the main result of this chapter.

Theorem 4.2. *For every $t > 0$ small enough, if G_t satisfies the Non-hyperbolicity hypothesis, then the map G_t has a non-hyperbolic invariant ergodic measure with an uncountable support.*

Theorem 4.2 follows from Propositions 4.3 and 4.5 presented below. First, Proposition 4.3 gives sufficient conditions for the existence of a non-hyperbolic ergodic measure. These conditions involve the existence of a sequence of periodic points with Lyapunov exponents going to zero (based in [DG09]). Second, Proposition 4.5 gives us the key ingredient for constructing the sequence of periodic orbits satisfying Proposition 4.3.

We will start with a periodic point of contracting type $P_t^{0,m}$, homoclinically related to P_t, and we construct another periodic point of contracting type with larger period and smaller absolute value of the Lyapunov exponent. Moreover, the second orbit is close to the one of initial point for a long time and away from the orbit of the initial point for a much shorter time (see Definition 4.3 and Figure 4.2). This allows to construct a sequence of periodic orbits of contracting type with Lyapunov exponents converging to zero. We use the fact that every point in a fundamental domain of $g_{0,t}$ has an expansive return (Lemma 4.1) and, since $g_{0,t}$ contracts in a neighborhood of p_t, we can control the expansion.

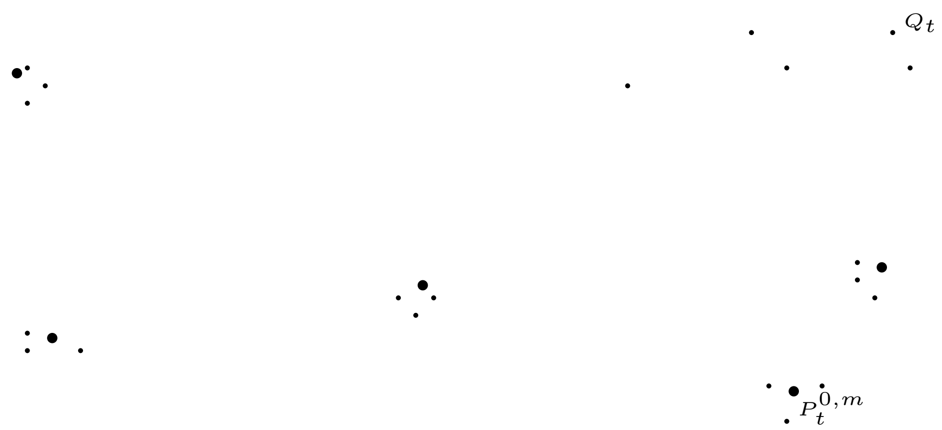

Figure 4.2: The orbit X_0 with initial point $P_t^{0,m}$ and the orbit X_1

For a given a finite set Υ, we denote by $\#\Upsilon$ the cardinality of Υ. We need the following definition.

Definition 4.3. [DG09, Definition 2.4] *Let* $\epsilon,\ \Theta > 0$*. A periodic orbit* Y *of a map* $f : M \to M$ *is a* (ϵ, Θ)-good approximation *of a periodic orbit* X *of* f *if the following properties hold:*

- *There exists a subset* Υ *of* Y *and a projection* $\rho : \Upsilon \to X$ *such that*

$$dist(f^i(y), f^i(\rho(y))) < \epsilon,\ \forall y \in \Upsilon,\ \forall i \in \{0, 1, \cdots, \pi(X) - 1\}$$

  where $\pi(X)$ *is the period of* X*.*

- $\frac{\#\Upsilon}{\#Y} \geq \Theta$*; and*
- $\#\rho^{-1}(x)$ *is the same for all* $x \in X$*.*

The next result appears in [DG09].

Proposition 4.3. [DG09, Proposition 2.5] *Assume that a diffeomorphism* $f : M \to M$ *has the following properties:*

1. *there exists an open domain* $O \subset M$ *such that* f *has an invariant continuous direction field* E *in* O*;*
2. *there exists a sequence of periodic orbits* $\{X_n\}_{n=1}^{\infty}$ *of* f *whose periods tend to infinity as* $n \to \infty$ *and such that* $\overline{\bigcup_{n=1}^{\infty} X_n} \subset O$*.*

Denote by $\lambda^E(X)$ *the Lyapunov exponent of* f *along the orbit* X *with respect to the invariant direction field* E*.*

3. *There exists a constant* $\xi \in (0, 1)$ *such that, for every* n, $|\lambda^E(X_{n+1})| < \xi|\lambda^E(X_n)|$*;*
4. *there exists a sequence of numbers* $\{\epsilon_n\}_{n=1}^{\infty}$, $\epsilon_n > 0$*, and a constant* $C > 0$ *such that for each* n *the orbit* X_{n+1} *is a* $(\epsilon_n, 1 - C|\lambda^E(X_n)|)$*- good approximation of the orbit* X_n*;*
5. *let* $\mathbf{d_n}$ *be the minimal distance between the points of the orbit* X_n*, then*

$$\epsilon_n < \frac{\min_{1 \leq i \leq n} \mathbf{d_i}}{3 \cdot 2^n}.$$

Then f *has a non-hyperbolic invariant ergodic measure with an uncountable support.*

The non-hyperbolic ergodic invariant measure obtained in the previous proposition has zero Lyapunov exponent along the direction E and is the limit measure of the sequence of measures uniformly distributed in the orbits X_n.

4.3 Lyapunov exponents converging to zero

In this section we construct the sequence of periodic orbits X_n (Proposition 4.5). For that we need to find $\delta_t > 0$ and $L_t > 1$ such that for every interval $J \subset D_t$, $0 < |J| < \delta_t$, there is a chain $\mathfrak{b} = \mathfrak{b}(J)$ satisfying

$$l_t|J| < |\Gamma_t^{\mathfrak{b}}(J)| < L_t|J|,$$

where l_t is given by the Non-hyperbolicity hypothesis. To get this property, we will make changes in the construction of the domain of $\Gamma_t^{\mathfrak{b}}$ for some chains $\mathfrak{b}$. The choice of the domain of definition of $\Gamma_t^{\mathfrak{b}}$ will play an important role in the proof of Proposition 4.5.

Since the conditions **(NH1)**, **(NH2)**, and **(NH3)** are open, there is a small $\gamma_t > 0$ such that:

- $\Gamma_t^{\mathfrak{b}^\star}(g_{0,t}(d_t) + \gamma_t) < g_{0,t}(d_t)$;
- $\left(\Gamma_t^{\mathfrak{b}^\star}\right)'(g_{0,t}(d_t) + \gamma_t) > l_t$; and
- $\left(\Gamma_t^{(u_i^\star+1,h)} \circ \Gamma_t^{(u_{i-1}^\star,h)} \circ \cdots \circ \Gamma_t^{(u_1^\star,h)}\right)'(\overline{d}_{t,\gamma_t}^i) > l_t$, where

$$\overline{d}_{t,\gamma_t}^i = \left(\Gamma_t^{(u_i^\star+1,h)} \circ \Gamma_t^{(u_{i-1}^\star,h)} \circ \cdots \circ \Gamma_t^{(u_1^\star,h)}\right)^{-1}(g_{0,t}(d_t) + \gamma_t),$$

  for each $i = 1, \ldots, j_0$.

On the other hand, from $(d_t, g_{0,t}(d_t)] = \bigcup_{u \geq u_1^\star} D_t^{(u,h)}$ and the monotonicity of $g_{0,t}$, there are $\overline{\mathfrak{u}} > \max_{1\leq i\leq j_0} u_i^\star$, where j_0 is defined in the Non-hyperbolicity hypothesis, and $e_t \in D_t^{\overline{\mathfrak{u}},h}$ (see Figure 4.5) such that

$$g_{0,t}(e_t) < g_{0,t}(d_t) + \gamma_t. \tag{4.3.1}$$

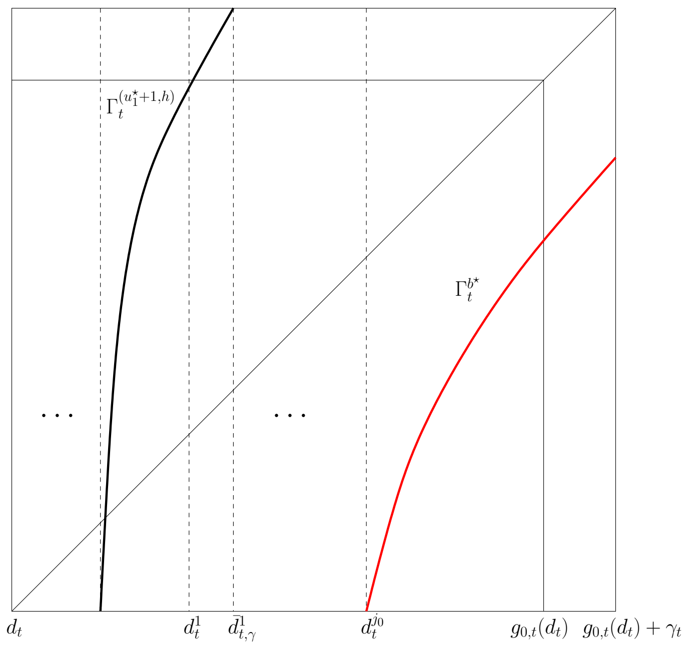

Figure 4.3: The definition of $\overline{d}^1_{t,\gamma_t}$ and the maps $\Gamma_t^{\mathfrak{b}^\star}$ and $\Gamma_t^{(u_1^\star+1,h)}$

Now we consider the following points: for each $i = 1, \dots, j_0$ and $u_i \geq u_1^\star$, we denote by (see Figure 4.4)

$$d_t^{i,u_i} := \left(\Gamma_t^{(u_i,h)} \circ \Gamma_t^{(u_{i-1}^\star,h)} \circ \cdots \circ \Gamma_t^{(u_1^\star,h)}\right)^{-1}(d_t)$$

and analogously, for each $i = 1, \dots, j_0$ and $u_i \geq u_i^\star$, we denote by

$$\overline{d}_{t,\gamma_t}^{i,u_i} := \left(\Gamma_t^{(u_i+1,h)} \circ \Gamma_t^{(u_{i-1}^\star,h)} \circ \cdots \circ \Gamma_t^{u_1^\star,h}\right)^{-1}(g_{0,t}(d_t)+\gamma_t).$$

From the construction above it is relevant to observe that $d_t^i = d_t^{(i,u_i^\star)}$ and $\overline{d}_{t,\gamma}^{i,u_i^\star} = \overline{d}_{t,\gamma}^i$. As

$$\left(d_t^{i-1}, g_{0,t}(d_t)\right] = \bigcup_{u_i \geq u_i^\star} D_t^{(u_1^\star,h)\cdots(u_i,h)},$$

and $d_t^{i,u_i+1} < d_t^{i,u_i}$, for all $u_i \geq u_i^\star$, we have that

$$d_t^{i,u_i} \to d_t^{i-1} \quad \text{as} \quad u_i \to +\infty,$$

where $d_t^0 := d_t$, for each $i = 1, \dots, j_0$ (see Figure 4.4).

In what follows, with a small abuse of notation, we can consider each function

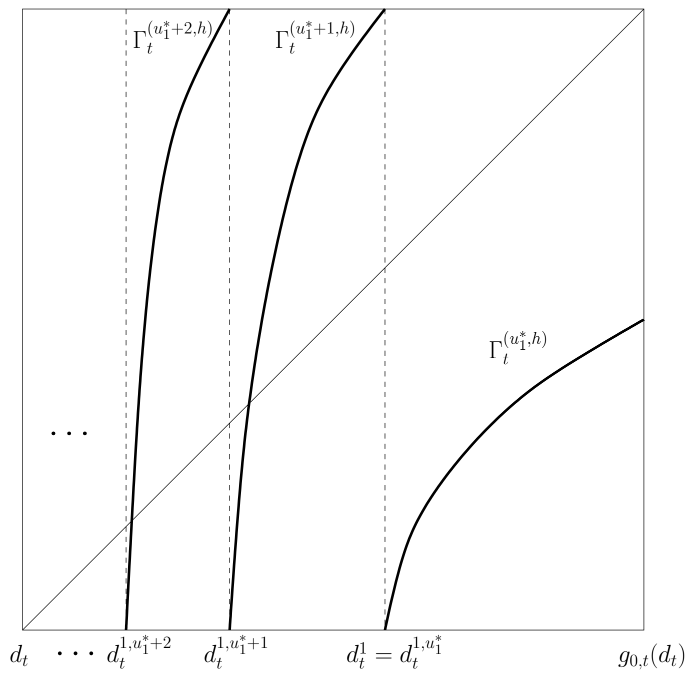

Figure 4.4: The definition of the points d_t^{1,u_1}

$\Gamma_t^{(u_1+1,h)}$ with extended domain as

$$\begin{array}{rcl}\Gamma_t^{(u_1+1,h)}: \ [d_t^{1,u_1+1}, \overline{d}_{t,\gamma_t}^{1,u_1}] & \to & (d_t, g_{0,t}(d_t)+\gamma_t] \\ y & \mapsto & \Gamma_t^{(u_1+1,h)}(y) = g_{0,t}^{u_1+1} \circ g_{[\omega_0\cdots\omega_{k_0}],t} \circ g_{0,t}^h(y),\end{array}$$

where $u_1 \geq u_1^\star$. We also consider $(d_t^{j_0}, g_{0,t}(d_t)+\gamma_t]$ as the domain of $\Gamma_t^{\mathfrak{b}^\star}$ and, for each $i = 2, \ldots, j_0$ and $u_i \geq u_i^\star$, we consider $[d_t^{i,u_i+1}, \overline{d}_{t,\gamma_t}^{i,u_i}]$ as the domain of $\Gamma_t^{\widetilde{\mathfrak{b}}_i}$, where

$$\widetilde{\mathfrak{b}}_i = (u_1^\star, h)\cdots(u_{i-1}^\star, h)(u_i+1, h).$$

From the definition of $\overline{d}_{t,\gamma_t}^i$ and since $g_{0,t}(e_t) < g_{0,t}(d_t)+\gamma_t$ we have, for each $i \in \{2, \ldots, j_0\}$,

$$\Gamma_t^{(u_i^\star,h)} \circ \cdots \circ \Gamma_t^{(u_1^\star,h)}(\overline{d}_{t,\gamma_t}^i) \ = \ g_{0,t}^{-1}(g_{0,t}(d_t)+\gamma_t) > e_t.$$

Thus, we can define (see Figure 4.5)

$$e_t^i := \left(\Gamma_t^{(u_i^\star,h)} \circ \cdots \circ \Gamma_t^{(u_1^\star,h)}\right)^{-1}(e_t) \in [d_t^i, \overline{d}_{t,\gamma_t}^i].$$

Finally, we can choose $\delta_t \in (0, \gamma_t)$ such that $g_{0,t}(e_t + \delta_t) < g_{0,t}(d_t) + \gamma_t$ and

$$\delta_t < \min_{i\in\{1,\dots,j_0\}} \left\{ |\overline{d}^{i}_{t,\gamma} - e^i_t|, |\overline{d}^{i,u_i}_{t,\gamma} - d^{i,u_i}_t| : u^\star_i < u_i \leq \overline{\mathfrak{u}} \right\}. \tag{4.3.2}$$

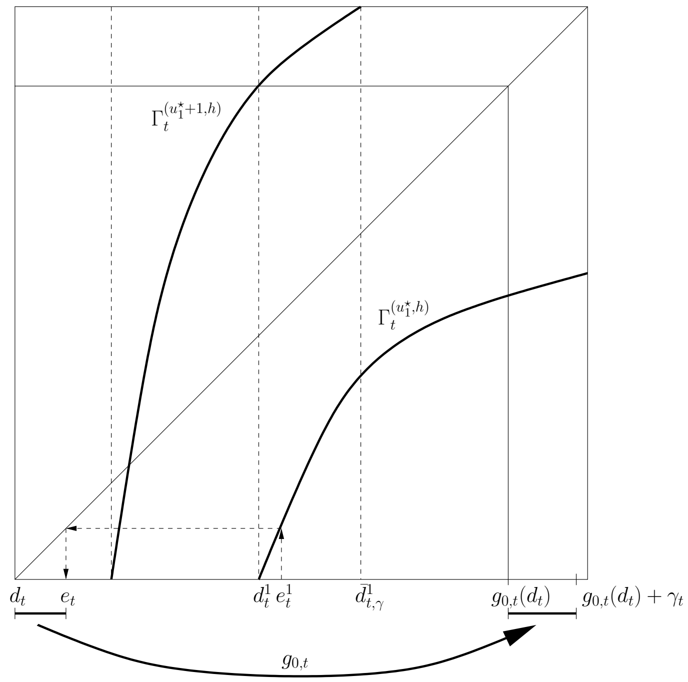

Figure 4.5: The definition of e^i_t

Lemma 4.4. *Assume that the Non-hyperbolicity hypothesis hold and consider an interval J with $J \subset (d_t, g_{0,t}(d_t)]$ and $0 < |J| < \delta_t$, δ_t as in (4.3.2). Then there is a chain $\mathfrak{b}$ such that $\Gamma^{\mathfrak{b}}_t(J) \subset (d_t, g_{0,t}(d_t) + \gamma_t]$ is also an interval and*

$$1 < l_t < \left(\Gamma^{\mathfrak{b}}_t\right)'(x) < L_t, \textit{for all } x \in J,$$

where $L_t := \left((\Gamma^{(\overline{\mathfrak{u}},h)}_t)'(e_t)\right)^{j_0}$ *and* j_0 *as in* **(NH)**.

Proof. Let J be an interval satisfying the assumptions of the lemma. Recall that $\mathfrak{b}^\star = (u^\star_1, h)\cdots(u^\star_{j_0}, h)$ and observe that, from the definitions of γ_t and L_t, we have

$$1 < l_t < \left(\Gamma^{\mathfrak{b}^\star}_t\right)'(x) < \left((\Gamma^{(\overline{\mathfrak{u}},h)}_t)'(e_t)\right)^{j_0} = L_t, \text{ for all } x \in [d^{j_0}_t, g_0(d_t) + \gamma]. \tag{4.3.3}$$

There are three possibilities: $J \subset [d_t, e_t + \delta_t]$, $J \subset [e_t, g_{0,t}(d_t)]$ and $J \cap D^{(u_1,h)}_t \neq \emptyset$

with $u_1^\star < u_1 < \overline{\mathfrak{u}}$, or $J \subset D_t^{(u_1^\star,h)}$.

Case 1. $J \subset [d_t, e_t + \delta_t]$
In this case we have $g_{0,t}(J) \subset [g_{0,t}(d_t), g_{0,t}(d_t)+\gamma]$ (see Figure 4.5). Once the derivative of $\Gamma_t^{\mathfrak{b}^\star}$ is decreasing, for each $x \in J$, one has

$$\begin{aligned}\left(\Gamma_t^{(u_{j_0}^\star,h)} \circ \cdots \circ \Gamma_t^{(u_1^\star,h+1)}\right)'(x) &= \left(\Gamma_t^{\mathfrak{b}^\star} \circ g_{0,t}\right)'(x) \\ &= \left(\Gamma_t^{\mathfrak{b}^\star}\right)'(g_{0,t}(x))\left(g_{0,t}'(x)\right) > l_t,\end{aligned}$$

where the last inequality follows from $g_{0,t}'(x) > 1$, for all $x \in J$, and by **(NH2)**.

On the other hand, using the definitions of $\overline{\mathfrak{u}}$ and L_t, one gets

$$\left(\Gamma_t^{\mathfrak{b}^\star} \circ g_{0,t}\right)'(x) < L_t, \ \forall x \in J.$$

Thus, the conclusion follows if we consider $\mathfrak{b} = (u_{j_0}^\star, h)\cdots(u_1^\star, h+1)$.

Case 2. $J \subset [e_t, g_{0,t}(d_t)]$ and $J \cap D_t^{(u_1,h)} \neq \emptyset$ with $u_1^\star < u_1 \leq \overline{\mathfrak{u}}$
There are two subcases. First, if $d_t^{1,u} \in J$ for some $u \in \{u_1 - 1, u_1\}$, then

$$J \subset [d_t^{1,u} - \delta_t, \overline{d}_{t,\gamma_t}^{1,u}],$$

since $|J| < \delta_t$ and, by (4.3.2), $|\overline{d}_{t,\gamma}^{1,u} - d_t^{1,u}| < \delta_t$. Moreover $\Gamma_t^{(u_1+1,h)}(J)$ is an interval and

$$l_t < \left(\Gamma_t^{(u+1,h)}\right)'(x) < \left(\Gamma_t^{(\overline{\mathfrak{u}},h)}\right)'(e_t) < L_t.$$

Second, if $d_t^{1,u} \notin J$, for all $u \in \{u_1 - 1, u_1\}$, then $J \subset D_t^{(u_1,h)}$, with $u_1^\star + 1 \leq u_1 \leq \overline{\mathfrak{u}}$, thus the claim follows from the monotonicity of $g_{0,t}'$ and the inequality $\left(\Gamma_t^{(u_1^\star+1,h)}\right)'(d_t^1) > l_t$, considering $\mathfrak{b} = (u_1, h)$.

Case 3. $J \subset D_t^{(u_1^\star,h)}$
Then either $J \subset D_t^{\mathfrak{b}^\star}$ or $J \cap [d_t^i, d_t^{i+1}] \neq \emptyset$ for some $i \in \{1, \ldots, j_0\}$. In the first case, $\Gamma_t^{\mathfrak{b}^\star}(J)$ is an interval and, by (4.3.3), $l_t < |\Gamma_t^{\mathfrak{b}^\star}(y)| < L_t$ holds, for all $y \in J$. In the other case the interval J must satisfy one of the following three possibilities:

(i) $J \subset [d_t^i - \delta_t, \overline{d}_{t,\gamma}^i]$,

(ii) $J \subset [e_t^i, d_t^{i+1}]$, and

(iii) $J \subset [d_t^{i+1} - \delta_t, \overline{d}_{t,\gamma}^{i+1}]$.

In the case **(i)**, from the condition **(NH)**, the claim holds with

$$\mathfrak{b} = (u_1^\star, h) \cdots (u_{i-1}^\star, h)(u_i^\star + 1, h).$$

In the case **(ii)**, by the definition of δ_t, we have $J \subset D_t^{\mathfrak{b}_i} \cup [d_t^{i,u_i}, \overline{d}_{t,\gamma}^{i,u_i}]$ with

$$\mathfrak{b}_i = (u_i + 1, h)(u_{i-1}^\star, h) \cdots (u_1^\star, h) \text{ and } u_i^\star \leq u_i < \mathfrak{u},$$

and consequently $\Gamma_t^{\mathfrak{b}_i}(J)$ is an interval and $l_t|J| < |\left(\Gamma_t^{\mathfrak{b}_i}\right)'(J)| < L_t|J|$ holds.

In the case **(iii)**, we get the claim with $\mathfrak{b} = (u_1^\star, h) \cdots (u_i^\star, h)(u_{i+1}^\star + 1, h)$. Now the proof of the lemma is completed. □

In order to construct a sequence of periodic orbits whose Lyapunov exponent along the fiber goes to zero, we follow the approach in [GIKN05]. Thus, in what follows we fix $m \in \mathbb{N}$ large enough such that

$$(\Gamma_t^{(0,m)})'(x) \leq 1,$$

for all $x \in (d_t, g_{0,t}(d_t) + \gamma_t]$ and $\Gamma_t^{(0,m)}((d_t, g_{0,t}(d_t) + \gamma_t]) \subset (d_t, g_{0,t}(d_t)]$. Consequently the map $\Gamma_t^{(0,m)}$ has an attracting fixed point $p_t^{0,m}$ (see Figure 4.6) which implies that the diffeomorphism G_t has a periodic point of contracting type, denoted by $P_t^{0,m} = \left((0^m\omega_0 \cdots \omega_{k_0})^{\mathbb{Z}}, p_t^{0,m}\right)$, of period $\pi = m + k_0 + 1$.

Proposition 4.5. *Suppose that the skew-product map G_t satisfies the Non-hyperbolicity hypothesis. If X_0 is a periodic orbit with Lyapunov exponent along the fiber λ_t and initial point $P_t^{0,m}$, then, for each $\epsilon > 0$, there exist a periodic orbit Y of G_t with Lyapunov exponent $\lambda_t' < 0$ in the central direction such that the following conditions hold:*

1. $|\lambda_t'| < |\lambda_t|(1 - \frac{\log l_t}{2 \log L_t})$, *where l_t is defined in condition* **(NH)** *and L_t in Lemma 4.4;*
2. *the periodic orbit Y is a $(2\epsilon, 1 - \frac{2|\lambda_t|}{\log L_t})-$ good approximation of the orbit of X_0.*

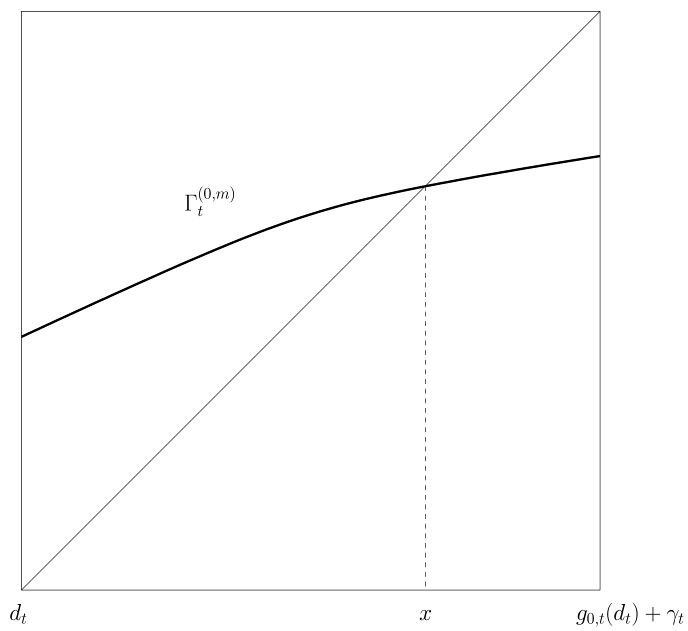

Figure 4.6: The point $p_t^{0,m}$ and the map $\Gamma_t^{(0,m)}$

Proof. Let X_0 be a periodic orbit of G_t satisfying the hypotheses of the proposition. Then $\lambda_t = \frac{\log \alpha}{\pi}$, where $\alpha := (\Gamma_t^{(0,m)})'(p_t^{0,m}) < 1$ is the Lyapunov exponent along the fiber of the orbit X_0. Now, we take constants α_- and α_+ close to α such that

$$0 < \alpha_- < \alpha < \alpha_+ < 1.$$

The constants α_- and α_+ will be fixed below.

For $\epsilon > 0$, there is a small interval $J = J(\alpha_-, \alpha_+) \subset (d_t, g_{0,t}(d_t) + \gamma]$ such that $p_t^{0,m} \in \text{int}(J)$,

$$\chi_t^{\pi}|J| < \epsilon, \text{ with } \chi_t = \max_{i \in \{0,\ldots,n-1\}} \left(\max_{x \in [q_t, p_t]} |g'_{i,t}(x)| \right), \tag{4.3.4}$$

and $\alpha_- \leq (\Gamma_t^{0,m})'(x) \leq \alpha_+$, for all $x \in J$. Recall that π is the period of $P_t^{0,m}$. Consequently, for all $r \in \mathbb{N}$, we have

$$\left| \left(\Gamma_t^{(0,m)} \right)^r (J) \right| \leq \alpha_+^r |J|. \tag{4.3.5}$$

Since, for all $y \in (d_t, g_{0,t}(d_t) + \delta_t]$, $(\Gamma_t^{(0,m)})^i(y) \to p_t^{0,m}$ as $i \to +\infty$, then we can define

$R_m(J)$ in the following way

$$R_m(J) := \min\{i \in \mathbb{N} : (\Gamma_t^{0,m})^i(g_{0,t}(d_t) + \delta_t) \in J \text{ and } (\Gamma_t^{0,m})^i(d_t) \in J\}$$

and we take

$$\varsigma_t := \min\{\delta_t, |J| L_t^{-R_m(J)}\}, \tag{4.3.6}$$

where δ_t is as in (4.3.2).

Since $(\alpha_+)^r \to 0$ as $r \to +\infty$, for r sufficiently large, there is a non-negative integer n_r such that $\varsigma_t / L_t \leq L_t^{n_r} \alpha_+^r |J| < \varsigma_t$ and, equivalently, we have

$$\frac{\varsigma_t}{L_t \alpha_+^r |J|} \leq L_t^{n_r} < \frac{\varsigma_t}{\alpha_+^r |J|}. \tag{4.3.7}$$

From $L_t^{n_r} \alpha_+^r |J| < \varsigma_t$, we conclude that $\alpha_+^r |J| < \varsigma_t < \delta_t$, hence, using the Lemma 4.4, there is a chain $\mathfrak{b}_1$ such that $\Gamma_t^{\mathfrak{b}_1}\left(\Gamma_t^{(0,m)}(J)\right)$ is an interval and

$$l_t < \left(\Gamma_t^{\mathfrak{b}_1}\right)' (\Gamma_t^{(0,m)}(J)) < L_t.$$

In the same way, since $L_t^{n_r} \alpha_+^r |J| < \delta_t$, then we can apply Lemma 4.4 n_r times to obtain n_r chains, $\mathfrak{b}_1, \ldots, \mathfrak{b}_{n_r}$, such that, for each $i = 1, \ldots, n_r$, the map $\Gamma_t^{\mathfrak{b}_i}$ expands at least by the factor l_t and less than the factor L_t and $\Gamma_t^{\mathfrak{b}_i} \circ \cdots \circ \Gamma_t^{\mathfrak{b}_1}(\Gamma_t^{(0,m)}(J))$ is an interval, for each $i \in \{1, \ldots, n_r\}$. Now, by notational reasons, we take

$$I = \Gamma_t^{\mathfrak{b}_{n_r}} \circ \cdots \circ \Gamma_t^{\mathfrak{b}_1} \circ (\Gamma_t^{0,m})^r(J)$$

and, by the definition of n_r, we have

$$|I| \leq L_t^{n_r} \alpha_+^r |J| < \varsigma_t. \tag{4.3.8}$$

As $I \subset J$, by definition of $R_m(J)$, we can find $\tilde{n} \leq R_m(J)$ such that $(\Gamma_t^{0,m})^{\tilde{n}}(I) \subset J$.

Taking

$$\Gamma_{t,r} := (\Gamma_t^{0,m})^{\tilde{n}} \circ \Gamma_t^{\mathfrak{b}_{n_r}} \circ \cdots \circ \Gamma_t^{\mathfrak{b}_1} \circ (\Gamma_t^{0,m})^r,$$

from (4.3.7), we conclude that

$$|\left(\Gamma_{t,r}\right)'(x)| < L_t^{\tilde{n}} L_t^{n_r} \alpha_+^r < L_t^{R_m(J)} \frac{\varsigma_t}{|J|} < 1$$

for all $x \in J$. Consequently, the map $\Gamma_{t,r}$ is a contraction on J and takes J into itself, hence it has a unique attracting fixed point $x' \in J$. For a chain $\mathfrak{b}$ recall the definition of $\theta(\mathfrak{b})$ given in (1.3.18). Now we consider the periodic point

$$X' = \left((0^m \omega_0 \cdots \omega_{k_0})^r \theta(\mathfrak{b}_1) \cdots \theta(\mathfrak{b}_{n_r}) (0^m \omega_0 \cdots \omega_{k_0})^{\tilde{n}})^{\mathbb{Z}}, x' \right) \in H_{\mathcal{V}}(P, G_t),$$

of contracting type, and its orbit Y with period $r\pi + \sum_{i=1}^{n_r} |\theta(\mathfrak{b}_i)| + \tilde{n}\pi$.

We claim that, for sufficiently large r,

$$\lambda_t' \geq \lambda_t \left(1 - \frac{\log l_t}{2 \log L_t}\right),$$

where λ_t' and λ_t are, respectively, the Lyapunov exponent along the fiber direction of Y and of X_0 (the periodic orbit with initial point $P_t^{0,m}$).

By definition of $\Gamma_{t,r}$ the constant $C_1 = C_1(\xi_m, J) := \left((\Gamma_t^{(0,m)})'(d_t) \right)^{R_m(J)}$, with $\xi_m = 0^m \omega_0 \cdots \omega_{k_0}$, verifies

$$|\left(\Gamma_{t,r}\right)'(x)| = |\left((\Gamma_t^{0,m})^{\tilde{n}} \circ \Gamma_t^{\mathfrak{b}_{n_r}} \circ \cdots \circ \Gamma_t^{\mathfrak{b}_1} \circ (\Gamma_t^{(0,m)})^r \right)(x)| > C_1 l_t^{n_r} \alpha_-^r \tag{4.3.9}$$

for all $x \in J$. By (4.3.7),

$$n_r \geq \frac{\log \varsigma_t - (\log L_t + r \log \alpha_+ + \log |J|)}{\log L_t}$$

so, the constant $C_2 = C_2(\xi_m, J) := \log \varsigma_t - \log L_t - \log |J|$ satisfies

$$n_r > \frac{1}{\log L_t} \left(-r \log \alpha_+ + C_2\right), \tag{4.3.10}$$

therefore, since $0 < \left(\Gamma_{t,r}\right)'(x') < 1$, from (4.3.9) and (4.3.10), we have

$$\begin{aligned} \log(\Gamma_{t,r})'(x') &\geq \log C_1 + r \log \alpha_- + n_r \log l_t \\ &> r \left(\log \alpha_- - \frac{\log l_t}{\log L_t} \log \alpha_+ \right) + \left(\frac{C_2 \log l_t}{\log L_t} + \log C_1 \right). \end{aligned} \tag{4.3.11}$$

Let $C_3 := C_2 \log l_t / \log L_t + \log C_1$. Now we specify the choice of $\alpha_- = (1-\tau)\alpha$ and $\alpha_+ = (1+\tau)\alpha$ by taking τ so close to zero such that the following inequality holds:

$$\log(1-\tau) - \frac{\log l_t}{\log L_t}\left(\log\left((1+\tau)\alpha\right)\right) \geq -\frac{\log l_t}{1.5 \cdot \log L_t}\log\alpha$$

$$\Leftrightarrow \quad \log\alpha_- - \frac{\log l_t}{\log L_t}\log\alpha_+ \geq \log\alpha\left(1 - \frac{\log l_t}{1.5\log L_t}\right). \tag{4.3.12}$$

As $\left(\Gamma_{t,r}\right)'(x') < 1$ and $r\pi + \sum_{i=1}^{n_r}|\theta(\mathfrak{b}_i)| + \tilde{n}\pi > r\pi$, the Lyapunov exponent of the orbit Y, λ_t', can be estimated as follows:

$$\lambda_t' \;=\; \frac{\log\left(\Gamma_{t,r}\right)'(x')}{r\pi + \sum_{i=1}^{n_r}|\theta(\mathfrak{b}_i)| + \tilde{n}\pi} > \frac{\log\left(\Gamma_{t,r}\right)'(x')}{r\pi},$$

thus, by (4.3.11) and (4.3.12), we get

$$\begin{aligned}\lambda_t' &\geq \frac{r\log\alpha\left(1 - \frac{\log l_t}{1.5\log L_t}\right) + C_3}{r\pi}\\ &= \lambda_t\left(1 - \frac{\log l_t}{1.5\log L_t}\right) + o\left(\tfrac{1}{r}\right),\end{aligned}$$

where $o\left(\frac{1}{r}\right) \to 0$ as $r \to \infty$. Consequently, for sufficiently large r, we have

$$\lambda_t' \geq \lambda_t\left(1 - \frac{\log l_t}{2\log L_t}\right).$$

and this proves the claim and thus the first item in Proposition 4.5.

Now, we claim that the periodic orbit Y is a $(2\epsilon, 1 - 2|\lambda_t|/\log L_t)$-good approximation of the orbit of X_0 (see Definition 4.3).

Considering $N = N(\epsilon, \xi_m)$ as the minimum integer such that $2^{-N\pi} < \epsilon$, we consider, for each $r > N$, the set

$$\Upsilon_r = \{G_t^i(X') \mid N\pi \leq i \leq (r-N-1)\pi\}.$$

For simplicity, in what follows, we omit the dependence on r, writing Υ instead of Υ_r.

The projection $\rho : \Upsilon \to X_0$ is defined by

$$\rho(G_t^i(X')) = G_t^d(P_t^{0,l}),$$

where $j = m\pi + d,\ 0 \le d < \pi$, that is d is the residue of division of j by π.

Now, to prove the claim, we need to verify the following conditions:

i) $d(G_t^i(\tilde{Y}), G_t^i(\rho(\tilde{Y}))) < 2\epsilon$, for every $\tilde{Y} \in \Upsilon$ and every $i = 0, 1, \dots, \pi - 1$;

ii) $\dfrac{\#\Upsilon}{\#Y} > 1 - \frac{2|\lambda_t|}{\log L_t}$;

iii) $\#\rho^{-1}(\tilde{X})$ is the same for all $\tilde{X} \in X_0$.

Consider $\tilde{Y} \in \Upsilon$ and $i \in \{0, 1, \dots, \pi - 1\}$. From the choice of N, the distance along the base, that is, the distance between the Σ_N-coordinates, of the points $G_t^i(\tilde{Y})$ and $G_t^i(\rho(\tilde{Y}))$ is less than ϵ.

It remains to estimate the distance along the fiber. For each $0 \le \kappa \le r$, the image of x' after $\kappa\pi$ iterations, which we denote by x'_κ, is contained in J and, from (4.3.4), we know that $\chi_t^\pi |J| < \epsilon$, we conclude that, after z iterations, $0 \le z < \pi$, the points $p_t^{0,m}$ and x'_κ cannot diverge by a distance greater that ϵ. Therefore, the orbits of $p_t^{0,m}$ and x' for the first $r\pi$ iterations diverge by a distance less than ϵ. Hence $d(G_t^i(\tilde{Y}), G_t^i(\rho(\tilde{Y}))) < 2\epsilon$ and the item i) is proved.

Now we prove the second item. We have

$$\begin{aligned}
1 - \frac{\#\Upsilon}{\#Y} &= \frac{(2N-1)\pi + n_r + \tilde{n}\pi}{r\pi + n_r + \tilde{n}} \\
&\le \frac{(2N-1)\pi + n_r + \tilde{n}\pi}{r\pi} \\
&\le \frac{C_4(\xi_m, J, \epsilon) + n_r}{r\pi},
\end{aligned}$$

where $C_4(\xi_m, J, \epsilon) = (2N-1)\pi + \tilde{n}\pi$. As $\alpha < \alpha_+$ and by (4.3.7),

$$n_r < \frac{\log \varsigma_t - \log |J| - r \log \alpha_+}{\log L_t} < \frac{\log \varsigma_t - \log |J| - r \log \alpha}{\log L_t},$$

then we conclude that

$$\begin{aligned} 1-\frac{\#\Upsilon}{\#Y} &\le \frac{C_5(\xi_m,J,\epsilon)}{r\pi}+\frac{-r\log\alpha}{r\pi\log L_t} \\ &= -\frac{\lambda_t}{\log L_t}+o\left(\frac{1}{r}\right), \text{ where } \lambda_t=\frac{\log\alpha}{\pi}. \end{aligned}$$

Thus, for sufficiently large r, we get

$$\frac{\#\Upsilon}{\#Y}>1-\frac{2|\lambda_t|}{\log L_t}.$$

Finally, to prove the item **iii)**, we just need to note that the number of points in $\rho^{-1}(\tilde{X})$, for all $\tilde{X}\in X_0$, is independent of $\tilde{X}$ and it is equal to $r-2N-1$. This completes the proof of Proposition 4.5. □

Proof of Theorem 4.2. Consider a small $\epsilon=\epsilon_0>0$, a point X_0 as in Proposition 4.5 and construct a periodic orbit $X_1=Y$ of contracting type. Applying inductively the Proposition 4.5 we can construct a sequence of periodic orbits X_ι and, in order to apply the Proposition 4.3, for each $\iota\in\mathbb{N}$, we define $\epsilon_\iota>0$. In fact, we set

$$\epsilon_{\iota+1}=\frac{\min_{1\le i\le\iota}\mathbf{d_i}}{3\cdot 2^\iota},$$

where $\mathbf{d_i}$, $i=1,\cdots,n$, is the minimum distance between two distinct point of X_ι. For the sequence ϵ_ι, the periods tend to infinity and the Lyapunov exponents along the fiber tend to zero, i.e., for all ι, at each step $\lambda_{\iota,t}\le 0$,

$$\lim_{\iota\to+\infty}|\lambda_{\iota,t}|\le|\lambda_t|\lim_{\iota\to+\infty}\left(1-\frac{\log l_t}{2\log L_t}\right)^\iota=0$$

Applying the Proposition 4.3 to the sequences $\{X_i\}_{i=0}^\infty$ we conclude that G_t has a non-hyperbolic invariant ergodic measure along the fiber with an uncountable support. □

Chapter 5

Hyperbolicity of the homoclinic classes

In this chapter, under some conditions on the global dynamics of the family $G_{a,t}$ and putting the fiber $\mathbb{S}^1$ instead of $[-1,1]$, we present a new family of skew-product maps $\left(\tilde{G}_{a,t}\right)_{t\in[-1,1]}$ unfolding a heterodimensional cycle at $t=0$.

We prove that, for $a > \log 4$, after the unfolding of the cycle and for a subset of the parameter space with positive relative density at the bifurcation value, the skew-product map $\tilde{G}_{a,t}$ is Ω-stable and the resulting non-wandering set, $\Omega(\tilde{G}_{a,t})'$, is the (disjoint) union of two hyperbolic basic sets, the homoclinic classes of P, $H(P,\tilde{G}_{a,t})$, and Q, $H(Q,\tilde{G}_{a,t})$.

We also derive similar results for the one-parameter family of diffeomorphisms $(f_{a,t})_{t\in[-1,1]}$ introduced in Section 2.4.

5.1 The model family of skew-products

In this section, we construct the model family $\tilde{G}_{a,t}$ and state the main result of this chapter.

Consider a skew-product map $\tilde{G}_{a,t}$ defined by

$$\tilde{G}_a : \Sigma_2 \times \mathbb{S}^1 \to \Sigma_2 \times \mathbb{S}^1, \quad \tilde{G}_a(\xi,x) = (\sigma(\xi), \tilde{g}_{\xi_0}(x)), \tag{5.1.1}$$

where $\tilde{g}_0 : \mathbb{S}^1 \to \mathbb{S}^1$ is a map with four hyperbolic fixed points, $0 < 1/2 < r_0 < a_0$, two repellers, 0 and r_0, and two attractors, 1/2 and a_0 (see Figure 5.1) such that

$$\tilde{g}_0(x) = \frac{2xe^a}{2xe^a + (1-2x)} = g_a(x), \ \forall x \in \left[0, \frac{1}{2}\right], \tag{5.1.2}$$

where g_a is the map introduced in 2.1.1, and $\tilde{g}_1 : \mathbb{S}^1 \to \mathbb{S}^1$ is a map with two hyperbolic fixed points, r_0 the repelling point and a_0 the attracting point, such that the following properties hold:

1. $\tilde{g}_1(1/2) = 0$,
2. $\tilde{g}_1([0, 1/2]) \subset [a_0, 1/2]$, and
3. $\tilde{g}_1^{-1}([0, 1/2]) \subset [1/2, r_0]$.

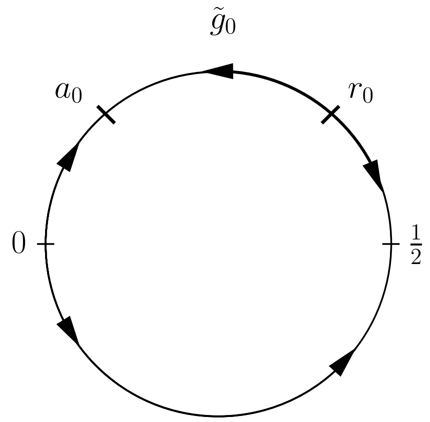

Figure 5.1: The map $\tilde{g}_0$

By the definition of (5.1.1), one has $\{0\}^{\mathbb{Z}} \times (0, 1/2) \subseteq W^s(P, \tilde{G}_a) \cap W^u(Q, \tilde{G}_a)$ and $A = (0^{-\mathbb{N}}.10^{\mathbb{N}}, 1/2) \in W^u(P, \tilde{G}_a) \cap W^s(Q, \tilde{G}_a)$, where $P = (0^{\mathbb{Z}}, 1/2)$ and $Q = (0^{\mathbb{Z}}, 0)$. Thus the skew-product map $\tilde{G}_a$ has a heterodimensional cycle associated to the fixed points P and Q. We also note that the non-wandering set of $\tilde{G}_a$, $\Omega(\tilde{G}_a)$, is given by

$$\Omega(\tilde{G}_a) = \Sigma_2 \times \{r_0, a_0\} \cup \{P, \ Q\}.$$

Now, for each $a > 0$, we define the one-parameter family of step skew-product maps $\left(\tilde{G}_{a,t}\right)_{t \in [-1,1]}$ by

$$\tilde{G}_{a,t} : \Sigma_2 \times \mathbb{S}^1 \to \Sigma_2 \times \mathbb{S}^1 \quad \tilde{G}_{a,t}(\xi, x) = (\sigma(\xi), \tilde{g}_{\xi_0,t}(x)),$$

where σ is the shift of two-symbols and $\tilde{g}_{0,t} = \tilde{g}_0$, for all $t > 0$. In what follows we assume that $\tilde{g}_{1,0} = \tilde{g}_1$ and that the map

$$\tilde{\mathfrak{g}}_1 : [-1,1] \to C^1(\mathbb{S}^1, \mathbb{S}^1), \quad \tilde{\mathfrak{g}}_1(t) = \tilde{g}_{1,t}$$

is continuous. We also suppose that $g_{1,t}$ is a C^1-map with respect to the variable t. Consequently, for each $t > 0$ small enough, $\tilde{g}_{1,t}$ has two fixed points, a repeller point r_t close to r_0 and an attractor point a_t close to a_0. Since a_0 and r_0 are fixed points of $\tilde{g}_0$ and the goal is to prove the Ω- stability of $\tilde{G}_{a,t}$, we need to assume that $a_t = a_0$ and $r_t = r_0$. In fact, if $a_t \neq a_0$ (or $r_t \neq r_0$), we claim that $\tilde{G}_{a,t}$ has a heterodimensional cycle associated to the fixed points $A_0 = (0^{\mathbb{Z}}, a_0)$ and $A_t = (1^{\mathbb{Z}}, a_t)$ ($R_0 = (0^{\mathbb{Z}}, r_0)$ and $R_{0,t} = (1^{\mathbb{Z}}, r_t)$ respectively).

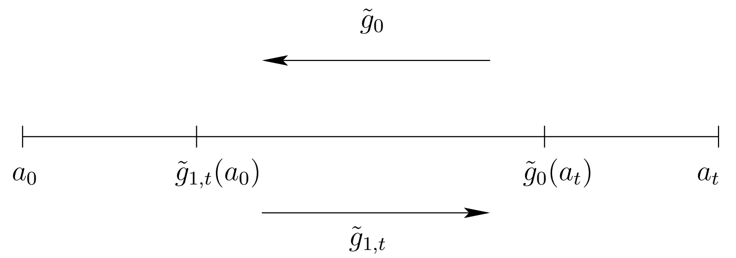

Figure 5.2: The points a_0 and a_t

In fact, to prove the claim, if we have $a_0 \neq a_t$, then

$$A^\star = \big(1^{-\mathbb{N}}0.0^{\mathbb{N}}, \tilde{g}_0(a_t)\big) \in W^s(A_0, \tilde{G}_{a,t}) \cap W^u(A_{0,t}, \tilde{G}_{0,t})$$

and

$$A^{\star\star} = \big(0^{-\mathbb{N}}1.1^{\mathbb{N}}, \tilde{g}_{1,t}(a_0)\big) \in W^u(A_0, \tilde{G}_{a,t}) \cap W^u(A_{0,t}, \tilde{G}_{0,t}),$$

consequently $W^s(A_0, \tilde{G}_{a,t}) \cap W^u(A_{0,t}, \tilde{G}_{0,t}) \neq \emptyset$ and $W^u(A_0, \tilde{G}_{a,t}) \cap W^s(A_{0,t}, \tilde{G}_{0,t}) \neq \emptyset$, therefore $\tilde{G}_{a,t}$ has a heterodimensional cycle. In a similar way we prove the claim for the case $r_0 \neq r_t$.

Assume that $\tilde{g}_{1,t}$ is affine in a neighborhood of $1/2$. As in Chapter 2, we can suppose that there is $\varepsilon > 0$ such that $\tilde{g}_{1,t}(x) = x - 1/2 + t$, for each $x \in [1/2 - \varepsilon, 1/2 + \varepsilon]$.

The objective of this chapter is to describe the dynamics of the skew-product maps $\tilde{G}_{a,t}$ for a large set of parameters t, that is, a set with positive relative density at the bifurcation $t = 0$.

We note that if $X \in \Sigma_2 \times (1/2, 1)$ and $X \notin \Sigma_2 \times \{a_0, r_0\}$, then X is wandering. Moreover, if $X \in \Sigma_2 \times [0, 1/2]$ and $\tilde{G}^i_{a,t}(X) \notin \Sigma_2 \times [0, 1/2]$ for some $i > 0$ (respectively $\tilde{G}^{-i}_{a,t}(X) \notin \Sigma_2 \times [0, 1/2]$ for some $i > 0$), then X is wandering and $X \in W^s(A_0, \tilde{G}_{a,t})$ (respectively $W^u(R_0, \tilde{G}_{a,t})$), consequently

$$\Omega(\tilde{G}_{a,t}) \subseteq \Sigma_2 \times \{r_0, a_0\} \bigcup \left(\bigcap_{i \in \mathbb{Z}} \tilde{G}^i_{a,t} \left(\Sigma_2 \times \left[0, \frac{1}{2}\right] \right) \right). \tag{5.1.3}$$

Define the maximal invariant set $\Lambda_{a,t} := \bigcap_{n \in \mathbb{Z}} \tilde{G}^i_{a,t} \left(\Sigma_2 \times [0, 1/2]\right)$; due to the existence of the filtration, it follows that

$$\Omega(\tilde{G}_{a,t})' = \Omega(\tilde{G}_{a,t}) \cap \Sigma_2 \times \left[0, \frac{1}{2}\right] \subseteq \Lambda_{a,t},$$

where, as we said before, $\Omega(\tilde{G}_{a,t})'$ is the resulting non-wandering set of $\tilde{G}_{a,t}$. Therefore, we can restrict our attention to the dynamics on the maximal invariant set of $\tilde{G}_{a,t}$ in $\Sigma_2 \times [0, 1/2]$.

Now we state the main result whose proof is postponed in the Section 5.3.

Theorem 5.1. *If $a > \log 4$, then there are $n_0 = n_0(a) \in \mathbb{N}$, $(t_n)_{n \geq n_0} = (t_n(a))_{n \geq n_0}$ a decreasing sequence converging to zero as $n \to +\infty$, and, for each $n \geq n_0$, a parameter $\mu^\star_{a,t_n} \in (t_{n+1}, t_n)$ such that:*

1. *$\tilde{G}_{a,t_n}$ has a heterodimensional cycle associated to saddles $P = (0^{\mathbb{Z}}, 1/2)$ and $Q = (0^{\mathbb{Z}}, 0)$, $\forall n \geq n_0$,*

2. *for every parameter $t \in \left(\mu^\star_{a,t_n}, t_n\right)$, the non-wandering set $\Omega(\tilde{G}_t)$ is hyperbolic, $\Omega(\tilde{G}_{a,t}) = \Sigma_2 \times \{r_0, a_0\} \cup H(P, \tilde{G}_{a,t}) \cup H(Q, \tilde{G}_{a,t})$ and*

$$\lim_{n \to \infty} \frac{t_n - \mu^\star_{a,t_n}}{t_n - t_{n+1}} > 0.$$

3. *$(\tilde{G}_{a,t})_{t \in [t_{n+1}, t_n]}$ has a saddle-node S_{a,t_n} at the parameter $\mu^\star_{a,t_n}$ and the intersection $H(P, \tilde{G}_{a,\mu^\star_{a,t_n}}) \cap H(Q, \tilde{G}_{a,\mu^\star_{a,t_n}})$ is exactly the orbit of S_{a,t_n},*

4. *moreover $\lim_{a \to +\infty} \left(\lim_{n \to \infty} \frac{t_n - \mu^\star_{a,t_n}}{t_n - t_{n+1}} \right) = 1.$*

We note that the parameters t_n are defined in Chapter 2 and correspond to secondary cycles, that is, parameters $t > 0$ such that $\tilde{G}_{a,t}$ has a cycle associated to P and Q. We also observe that if we consider $a > \log 4$, the hyperbolicity is prevalent, but not totally prevalent. However, since $\lim_{n\to\infty} t_{n+1}/t_n = e^{-a/2}$, $e^{-a/2} \to 0$ as $a \to +\infty$, and $\lim_{a\to+\infty} \big(\lim_{n\to\infty}(t_n - \mu^{\star}_{a,t_n})/(t_n - t_{n+1})\big) = 1$, the frequency of hyperbolicity becomes close to one, for large a, that is, fixed any ϵ, there is a large a^+ such that for $a > a^+$,

$$\liminf_{t\to 0^+} \frac{|\mathbb{H}(t)|}{t} > 1 - \epsilon$$

where $\mathbb{H}(s)$ is the set of parameters $t \in (0, s)$ such that $\Omega(\tilde{G}_{a,t})'$ is hyperbolic.

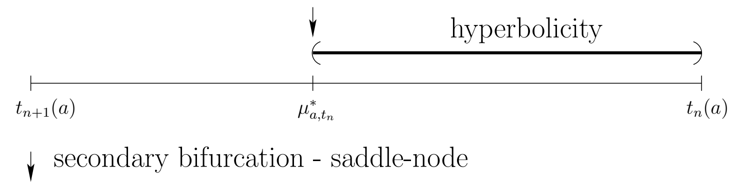

Figure 5.3: The hyperbolic parameter intervals

The proof of Theorem 5.1 is presented in Section 5.3 however, in the next section, we give an idea of it after the introduction of a system $\tilde{\mathfrak{G}}_{a,t}$ of iterated functions generated by $\tilde{g}_0$ and $\tilde{g}_{1,t}$.

5.2 Hyperbolic systems of iterated functions

In this section, we consider the system of iterate functions generated by $\tilde{g}_0$ and $\tilde{g}_{1,t}$, $\tilde{\mathfrak{G}}_{a,t}$, whose precise definition is given below, and for which we obtain hyperbolicity-like properties that will be translated into hyperbolic dynamics of the maps $\tilde{G}_{a,t}$ and $f_{a,t}$.

In what follows we consider $a > \log 4$ and, as in Section 2.2, $t_n = t_n(a)$ is given by $g_a^n(t_n) = 1/2 - t_n$, for $n \in \mathbb{N}$ large. For each $t \in (t_{n+1}, t_n)$, consider the fundamental domain of g_a,

$$D_{a,t} := [d_{a,t}, g_a(d_{a,t})],$$

where $d_{a,t} = g_a^{-n}(1/2 - t)$, and define $\tilde{\Delta}_{a,t} := \Sigma_2 \times D_{a,t}$, see (5.1.2).

We observe that if $X \in \Lambda_{a,t} \cap \Omega(\tilde{G}_{a,t})'$ and $X \notin \{0^{\mathbb{Z}}\} \times [0, 1/2]$, then there is $i \in \mathbb{Z}$ such that $\tilde{G}^i_{a,t}(X) \in \tilde{\Delta}_{a,t}$, that is, X has some iterate in $\tilde{\Delta}_{a,t}$. If we consider a point

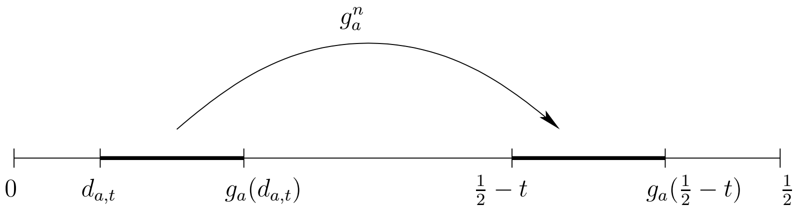

Figure 5.4: The transition map

$X=(\xi,x)\in\Lambda_{a,t}\cap\tilde{\Delta}_{a,t}$ such that $X\notin W^s(P,\tilde{G}_{a,t})\cup W^s(Q,\tilde{G}_{a,t})$, then there is a first $j_1>1$ such that $\tilde{G}^{j_1}_{a,t}(X)\in\tilde{\Delta}_{a,t}$ and, by definition of $\tilde{G}_{a,t}$, one has $j_1=n+r_1+1+u_1$ and

$$\tilde{G}^{j_1}_{a,t}(X)=(\cdots\xi_{-1}0^{n+r_1}10^{u_1}.\xi_{j_1}\cdots,\tilde{g}_{[0^{n+r_1}10^{u_1}]}(x)) \tag{5.2.1}$$

where recall $g_a^n(D_{a,t})=[1/2-t,g_a(1/2-t)]$. Similar considerations can be made for the backward orbit of any point

$$X\in\Lambda_{a,t}\cap\tilde{\Delta}_{a,t}$$

such that $X\notin W^u(P,\tilde{G}_{a,t})\cup W^u(Q,\tilde{G}_{a,t})$.

The equation (5.2.1) leads to the following one-parameter family of maps describing the returns to $\tilde{\Delta}_{a,t}$:

$$\tilde{\Gamma}^{(u,r)}_{a,t}:D^{(u,r)}_{a,t}\to D_{a,t},\quad \tilde{\Gamma}^{(u,r)}_{a,t}(x)=g_{[0^{n+r}10^u]}(y)=g_a^u\circ\tilde{g}_{1,t}\circ g_a^{n+r}(x), \tag{5.2.2}$$

where $D^{(u,r)}_{a,t}$ is the maximal subset of $D_{a,t}$ which contains the points y such that $\tilde{\Gamma}^{(u,r)}_{a,t}(y)\in D_{a,t}$. Now we define the system of iterated functions generated by $\tilde{g}_0$ and $\tilde{g}_{1,t}$ as the set:

$$\tilde{\mathfrak{G}}_{a,t}=\{\tilde{\Gamma}^{(u,r)}_{a,t}:(u,r)\in\mathbb{N}_0\times\mathbb{N}_0\}.$$

Let $a>\log 4$. The idea of the proof of Theorem 5.1 is the following. First we prove that the arc $(\tilde{\Gamma}^{(1,0)}_{a,t})_t$ has a saddle-node for the parameter $\mu^\star_{a,t_n}\in(t_{n+1},t_n)$. For $t>\mu^\star_{a,t_n}$, the map $\tilde{\Gamma}^{(1,0)}_{a,t}$ has two fixed points, $s^-_{a,t}$ (repelling) and $s^+_{a,t}$ (attracting), collapsing to the saddle-node $s_{a,\mu^\star_{a,t_n}}$ at $\mu^\star_{a,t_n}$. Therefore, for $t\in(\mu^\star_{a,t_n},t_n)$, these points corresponds to periodic points $S^-_{a,t}=((0^n10)^{\mathbb{Z}},s^-_{a,t})$ and $S^+_{a,t}=((0^n10)^{\mathbb{Z}},s^+_{a,t})$ of expanding and contracting type, respectively. Afterwards, we get hyperbolic properties for the system $\tilde{\mathfrak{G}}_{a,t}$. Next we translate these properties to the skew-product map G_t and we prove that for, $t \in (\mu^\star_{a,t_n}, t_n)$, we have

$$H(P, G_t) = H(S^+_{a,t}, G_t) \text{ and } H(P, G_t) = H(S^+_{a,t}, G_t).$$

These classes are both hyperbolic and their union is the resulting non-wandering set.

In what follows we obtain the properties for the system of iterated functions $\tilde{\mathfrak{G}}_{a,t}$. First note that, for each $t \in (t_{n+1}, t_n)$, we have

$$g_a^n(t) < g_a^n(t_n) = \frac{1}{2} - t_n < \frac{1}{2} - t,$$

therefore $\tilde{g}_{1,t}(1/2) = t < d_{a,t}$ and consequently, $D^{(0,r)}_{a,t} = \emptyset$, for all $r \geq 0$. Moreover, for each $t \in (t_{n+1}, t_n)$, we also have

$$g_a^n(g_a(t)) = g_a^{n+1}(t) \geq g_a^{n+1}(t_{n+1}) = 1/2 - t_{n+1} > 1/2 - t,$$

which implies that $g_a(t) \in D_{a,t}$.

Proposition 5.2. *For $a > \log 4$ there is a large $n_0 = n_0(a)$ such that, for all $n \geq n_0$, there is $\mu^\star_{a,t_n} \in (t_{n+1}, t_n)$ such that, the map $\tilde{\Gamma}^{(1,0)}_{a,t}$ has two hyperbolic fixed points in $D_{a,t}$, $s^-_{a,t} < s^+_{a,t}$, for every $t \in (\mu^\star_{a,t}, t_n)$. These points collapse to the saddle node s_{a,t_n} at $t = \mu^\star_{a,t_n}$ and disappear for $t \in (t_{n+1}, \mu^\star_{a,t_n})$.*

Proof. Arguing as in Proposition 2.12, one has that the map $g_{[0^n10],t}$ has a saddle node at

$$t = \frac{2t_n}{(1 - 2t_n)e^{\frac{a}{2}} + 2t_n} = \mu^\star_{a,t_n} \tag{5.2.3}$$

and, since $a > \log 4$, we have $\mu^\star_{a,t_n} < t_n$ for n large. Recall that

$$t_{n+1} = \frac{t_n}{(1 - 2t_n)e^{a/2} + 2t_n}, \tag{5.2.4}$$

thus we also have $t_{n+1} < \mu^\star_{a,t_n}$. From the monotonicity of the map

$$\mathbf{\Gamma} : (t_{n+1}, t_n] \to D_{a,t}, \quad t \mapsto \tilde{\Gamma}^{(1,0)}_{a,t},$$

we can conclude that the map $\tilde{\Gamma}^{(1,0)}_{a,t}$ has a pair of fixed points, $s^-_{a,t}$ and $s^+_{a,t}$, where $s^-_{a,t}$ is expanding and $s^+_{a,t}$ is contracting, for $t \in (\mu^\star_{a,t_n}, t_n)$ and, for $t \in (t_{n+1}, \mu^\star_{a,t_n})$, the map $\tilde{\Gamma}^{(1,0)}_{a,t}$ is below the diagonal. □

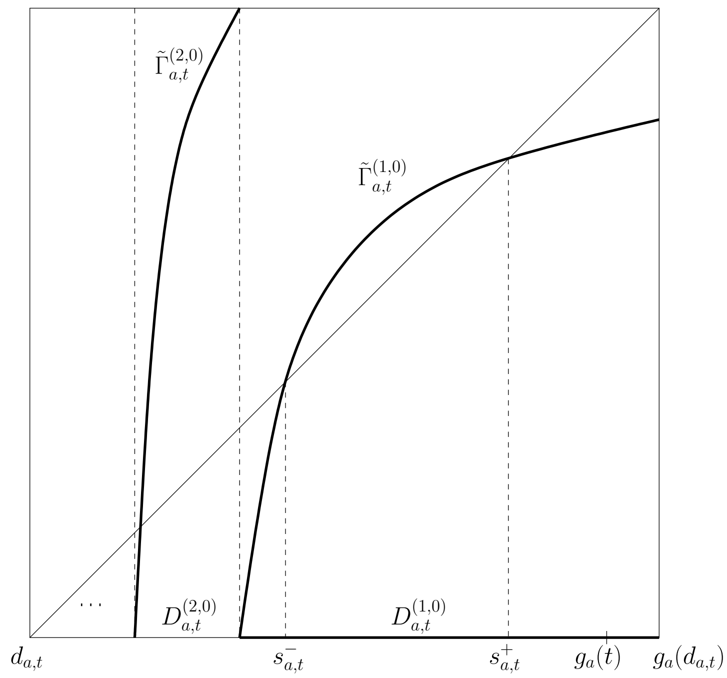

Figure 5.5: The maps $\tilde{\Gamma}^{u,0}_{a,t}$ and the sets $D^{u,0}_{a,t}$

Now, for $t \in \left(\mu^\star_{t_n,a}, t_n\right)$, consider the partition of $D_{a,t} = [d_{a,t}, g_a(d_{a,t})]$ given by the intervals

$$L_u(a,t) = \left[d_{a,t}, s^-_{a,t}\right], \quad L_c(a,t) = \left(s^-_{a,t}, s^+_{a,t}\right) \text{ and } L_s(a,t) = \left[s^+_{a,t}, g_a(d_{a,t})\right].$$

Before analyze the system $\tilde{\mathfrak{G}}_{a,t}$, we need to state the following definition.

Definition 5.1. *We say that a point x is* non-wandering *for $\tilde{\mathfrak{G}}_{a,t} = \{\tilde{\Gamma}^{(u,r)}_{a,t}\}$ if for any neighborhood U of x there are $n > 0$ and a chain $\mathfrak{b}$ such that $\tilde{\Gamma}^{\mathfrak{b}}_{a,t}(U)$ intersects U. The set of all non-wandering points for $\tilde{\mathfrak{G}}_{a,t}$ is called the non-wandering set of $\tilde{\mathfrak{G}}_{a,t}$ and is denoted by $\Omega(\tilde{\Gamma}^{(u,r)}_{a,t})$.*

The next lemmas, whose proofs are done using the techniques in [DR07], imply that the non-wandering set $\Omega(\tilde{\Gamma}^{(u,r)}_{a,t})$ is contained in $L_s(a,t) \cup L_u(a,t)$, for each $t \in \left(\mu^\star_{a,t_n}, t_n\right)$. Now, the conclusion of $\Omega(\tilde{\Gamma}^{(u,v)}_{a,t})$ hyperbolic comes from the fact that, for every $(u,r) \in \mathbb{N}_0 \times \mathbb{N}_0$ such that $D_{a,t}^{(u,r)} \cap L_s(a,t) \neq \emptyset$, respectively $D_{a,t}^{(u,r)} \cap L_u(a,t) \neq \emptyset$, the map $\tilde{\Gamma}_{a,t}^{(u,r)}$ is uniformly contracting in $L_s(a,t)$, respectively expanding in $L_u(a,t)$.

Lemma 5.3. *For n large and $t \in \left(\mu^\star_{a,t_n}, t_n\right)$, the following properties hold:*

1. *the restriction of $\tilde{\Gamma}_{a,t}^{(1,r)}$ to $L_s(a,t)$ is uniformly contracting and*

$$\tilde{\Gamma}_{a,t}^{(1,r)}(L_s(a,t)) \subset L_s(a,t), \quad \text{for all } r \geq 0;$$

2. *all the returns of $L_s(a,t)$ to $D_{a,t} = [d_{a,t}, g_a(d_{a,t})]$ are of the form $(1,r)$ with $r \geq 0$.*

Proof. Since $\tilde{\Gamma}_{a,t}^{(1,0)}$ is an increasing map and $s^+_{a,t}$ is a fixed point, we have $\tilde{\Gamma}_{a,t}^{(1,0)}(L_s(a,t)) \subseteq L_s(a,t)$ and

$$\tilde{\Gamma}_{a,t}^{(1,r)}(L_s(a,t)) \subseteq [\tilde{\Gamma}_{a,t}^{(1,0)}(g_a(d_{a,t})), g_a(t)] \subseteq L_s(a,t), \quad \text{for all } r \in \mathbb{N}.$$

Moreover, for all $x \in L_s(a,t)$,

$$0 < (\tilde{\Gamma}_{(a,t)}^{(1,r)})'(x) < \left(\tilde{\Gamma}_{a,t}^{(1,r)}\right)'(s^+_{a,t}) < \left(\tilde{\Gamma}_{a,t}^{(1,0)}\right)'(s^+_{a,t}) < 1, \ t \in \left(\mu^\star_{a,t_n}), t_n\right),$$

and the first item holds.

The second item follows from $\tilde{\Gamma}_{a,t}^{(1,r)}(L_s(a,t)) \subseteq L_s(a,t)$, hence, for all $u \geq 2$ and $x \in L_s(a,t)$, we have

$$\tilde{\Gamma}_{a,t}^{(u,r)}(x) = g_a^{u-1}(\tilde{\Gamma}_{a,t}^{(1,r)}(x)) > g_a^{u-1}(s^+_{a,t}) > g_a(d_{a,t})$$

and $\tilde{\Gamma}_{a,t}^{(u,r)}(x) \notin [d_{a,t}, g_a(d_{a,t})]$. □

Lemma 5.4. *For every $t \in \left(\mu^\star_{a,t_n}, t_n\right)$, the following properties hold:*

1. *$\tilde{\Gamma}_{a,t}^{(1,0)}(L_c(a,t)) = L_c(a,t)$ and the restriction of $\tilde{\Gamma}_{a,t}^{(1,0)}$ to $L_c(a,t)$ is strictly increasing;*

2. *$\tilde{\Gamma}_{a,t}^{(1,r)}(L_c(a,t)) \subseteq L_s(a,t), \forall r \geq 1$;*

3. *All return of points of $L_c(a,t)$ to $L_c(a,t)$ are of the form $(1,0)$.*

Proof. The first item of the lemma follows directly from the definitions of the map $\tilde{\Gamma}_{a,t}^{(1,0)}$ and the set $L_c(a,t)$. For the second item, observe that

$$\tilde{\Gamma}_{a,t}^{(1,r)}(L_c(a,t)) \subseteq \left[\tilde{\Gamma}_{a,t}^{(1,0)}(g_a(d_t)), g_a(t)\right] \subseteq L_s(a,t), \quad \text{for all } r \geq 1.$$

The last item follows from the two items above and the proof of Lemma 5.3. □

Lemma 5.5. *For every $t \in \left(\mu^\star_{a,t_n}, t_n\right)$, the following properties hold:*

1. $\tilde{\Gamma}_{a,t}^{(1,r)}(L_u(a,t)) \subset L_s(a,t),\ \forall r \geq 1$;
2. *Any return of points $L_u(a,t)$ to $L_u(a,t)$ are of the form $(u,0)$ with $u \geq 1$ and the restriction of $\tilde{\Gamma}_{a,t}^{(u,0)}$ to $L_u(a,t)$ is uniformly expanding for all $u \in \mathbb{N}$.*

Proof. For the first item, we observe that, if $r \geq 1$, then $\tilde{\Gamma}_{a,t}^{(1,r)}(L_u(a,t))$ is at the right of $\tilde{\Gamma}_{a,t}^{(1,0)}(L_s(a,t)) \subseteq L_s(a,t)$. Since $(\tilde{\Gamma}_{a,t}^{(1,0)})'(s^-_{a,t}) > 1$ and g'_a is a decreasing map, we obtain the expansivity of $\Gamma_{a,t}^{(\tilde{u},0)}$. □

5.3 Prevalent hyperbolicity of the model family of skew-products

As we referred before, in this section we prove the Theorem 5.1. To do that, for $t \in \left(\mu^\star_{a,t_n}, t_n\right)$, where $\mu^\star_{a,t_n}$ is the saddle-node parameter of the family $\tilde{\Gamma}_{a,t}^{(1,0)}$ considered in the previous section, we translate the hyperbolicity of the system $\left(\tilde{\Gamma}_{a,t}^{(u,r)}\right)$ to the skew-product maps $\tilde{G}_{a,t}$.

The next definition and the next lemma are just a reformulation of Definition 1.7 and Lemma 1.1, respectively.

Definition 5.2. *Given $X \in \Lambda_{a,t}$, we define the sequence $\{\varrho_j(X)\}_{j\in I(X)}$, $\varrho_j(X) \in \mathbb{Z}$, and the set $I(X) \subset \mathbb{Z}$ associated with X by*

- *If $j_1,\ j_2 \in I(X)$, then $j \in I(X)$, for all $j_1 \leq j \leq j_2$;*
- *$\varrho_j(X) < \varrho_{j+1}(X)$, for every $j,\ j+1 \in I(X)$;*
- *$\tilde{G}_{a,t}^{\varrho_j(X)}(X) \in \tilde{\Delta}_{a,t}$ and*
- *$\tilde{G}_{a,t}^k(X) \in \Lambda_{a,t} \setminus \tilde{\Delta}_{a,t}$, for every $\varrho_j(X) < k < \varrho_{j+1}(X)$.*

If $I(X) \neq \emptyset$, then we assume that $0 \in I(X)$. In this situation, if X has a positive iterate in $\tilde{\Delta}_{a,t}$, $\varrho_0(X)$ is the first positive iterate of X in $\tilde{\Delta}_{a,t}$, otherwise, $\varrho_0(X)$ is the first backward iterate of X in $\tilde{\Delta}_{a,t}$.

Lemma 5.6. *Consider $t > 0$ small and $X \in \Lambda_{a,t}$. Then:*

- *$I(X) = \emptyset$ if and only if $X \in \{0\}^{\mathbb{Z}} \times (0, 1/2)$;*
- *$I(X)$ is upper bounded if and only if $X \in W^s(P, \tilde{G}_{a,t}) \cup W^s(Q, \tilde{G}_{a,t})$;*
- *$I(X)$ is lower bounded if and only if $X \in W^u(P, \tilde{G}_{a,t}) \cup W^u(Q, \tilde{G}_{a,t})$.*

Definition 5.3. *Take $X \in \Lambda_{a,t}$ with sequence $\{\varrho_j(X)\}_{j \in I(X)}$. We define* the itinerary of the point X *as the sequence $\{i_j(X)\}_{j \in I(X)}$, $i_j(X) \in \{s, c, u\}$, such that*

$$i_j(X) = k \quad \text{if and only if} \quad \tilde{G}_{a,t}^{\varrho_j(X)}(X) \in \Sigma_2 \times L_k(a,t), \quad k = s, c, u.$$

For every $a > \log 4$ and $t \in (\mu^{\star}_{a,t_n}, t_n)$, with n large enough, consider the sets

$$\begin{aligned}
\Omega_s(a,t) &= \{X \in \Omega(\tilde{G}_{a,t})' : I(X) \neq \emptyset \text{ and } i_k(X) = s, \quad \text{for all } k \in I(X)\} \cup \{P\};\\
\Omega_c(a,t) &= \{X \in \Omega(\tilde{G}_{a,t})' : I(X) \neq \emptyset \text{ and } i_k(X) = c, \quad \text{for all } k \in I(X)\}; \textit{and}\\
\Omega_u(a,t) &= \{X \in \Omega(\tilde{G}_{a,t})' : I(X) \neq \emptyset \text{ and } i_k(X) = u, \quad \text{for all } k \in I(X)\} \cup \{Q\}.
\end{aligned}$$

Note that if $X \in \Lambda_{a,t}$ and $i_k(X) = c$ for all $k \in I(X)$, then all the returns to $\tilde{\Delta}_{a,t}$ are of the form $(1, 0)$ and the map restricted to $L_c(a,t)$ is increasing without fixed points. Thus, $\Omega_c(a,t) = \emptyset$.

From the definitions of the sets $\Omega_s(a,t)$ (respectively $\Omega_u(a,t)$) and $D_{a,t}$, we conclude that the sets $\Omega_s(a,t)$ and $\Omega_u(a,t)$ are $\tilde{G}_{a,t}-$ invariant. Moreover, all these sets are closed. To see, for instance, that $\Omega_s(a,t)$ is closed, take a sequence $\{X_n\}_{n \in \mathbb{N}}$ in $\Omega_s(a,t)$, with $X_n \to Z$ for some Z. Since $\Omega(\tilde{G}_{a,t})$ is a closed set, one has $Z \in \Omega(\tilde{G}_{a,t})$. Without loss of generality we can assume that $X_n \neq P$. If $Z = P$, then the result is obvious, otherwise, by replacing the sequence by some iterate of it by $\tilde{G}_{a,t}$, we can assume that $X_n \in \tilde{\Delta}_{a,t}$ for every n large enough, thus $I(Z) \neq \emptyset$. Since $X_n \in \Omega_s(a,t)$ for every n, we conclude that the itinerary of Z consist only of s, that is, $Z \in \Omega_s(a,t)$.

The following steps imply the second assertion of Theorem 5.1 and are inspired in [DR97].

Step A. Let $X \in \Lambda_{a,t}$.

(1) If there is $j \in I(X)$ such that $i_j(X) = s$, then, for all $k \in I(X) \cap (j, +\infty)$, $i_k(X) = s$.

(2) If there is $j \in I(X)$ such that $i_j(X) = u$, then, for all $k \in I(X) \cap (-\infty, j)$, $i_k(X) = u$.

Step B. $W^u(P, \tilde{G}_{a,t}) \cap \Omega(\tilde{G}_{a,t})' \subseteq \Omega_s(a,t)$ and $W^s(Q, \tilde{G}_{a,t}) \cap \Omega(\tilde{G}_{a,t})' \subseteq \Omega_u(a,t)$. Therefore, $H(P, \tilde{G}_{a,t}) \subseteq \Omega_s(a,t)$ and $H(Q, \tilde{G}_{a,t}) \subseteq \Omega_u(a,t)$.

Step C. One has:

(1) $W^u(P, \tilde{G}_{a,t}) \cap W^s(Q, \tilde{G}_{a,t}) = \emptyset$,

(2) $(W^s(P, \tilde{G}_{a,t}) \cap W^u(Q, \tilde{G}_{a,t}) \cap \Omega(\tilde{G}_{a,t})' = \emptyset$,

(3) $W^s(P, \tilde{G}_{a,t}) \cap W^u(P, \tilde{G}_{a,t}) = W^s(P, \tilde{G}_{a,t}) \pitchfork W^u(P, \tilde{G}_{a,t}) \subseteq H(P, \tilde{G}_{a,t})$, and

(4) $W^s(Q, \tilde{G}_{a,t}) \cap W^u(Q, \tilde{G}_{a,t}) = W^s(Q, \tilde{G}_{a,t}) \pitchfork W^u(Q, \tilde{G}_{a,t}) \subseteq H(Q, \tilde{G}_{a,t})$.

Step D. $\Omega(\tilde{G}_{a,t})' \cap \tilde{\Delta}_{a,t} \subseteq \Omega_u(a,t) \cup \Omega_s(a,t)$.

Step E. $\Omega_s(a,t) = H(P, \tilde{G}_{a,t})$, $\Omega_u(a,t) = H(Q, \tilde{G}_{a,t})$ and

$$\Omega(\tilde{G}_{a,t})' = \Omega_s(a,t) \cup \Omega_u(a,t) = H(P, \tilde{G}_{a,t}) \cup H(Q, \tilde{G}_{a,t}).$$

Step F. $\Omega_s(a,t)$ and $\Omega_u(a,t)$ are hyperbolic basic sets.

Step G. The skew-product $\tilde{G}_{a,t}$ has no cycles related to $\Omega_s(a,t)$ and $\Omega_u(a,t)$

Now we prove the steps A-G.

Proof of step A. By hypothesis, we have

$$G_t^{\varrho_j(X)}(X) \in \Sigma_2 \times L_s(a,t),$$

and, by Lemma 5.3, we get $i_k(X) = s$, for every $k \in I(X) \cap [j, +\infty]$, and the condition **(1)** holds. The condition **(2)** follows in similar way.

Proof of step B. Let $X \in W^u(P,\tilde{G}_{a,t}) \cap \Omega(\tilde{G}_{a,t})' \subseteq \Lambda_{a,t}$. Assuming that $X \neq P$, the situation $X = P$ is trivial, we have

$$X = (0^{-\mathbb{N}}10\xi_{-m}\cdots\xi_{-1}.\xi_0\cdots, x), \quad \text{with } x = \tilde{g}_{[10\xi_{-m}\cdots\xi_{-1}],t}\left(\frac{1}{2}\right),$$

and

$$Z = \tilde{G}_{a,t}^{-(m+1)}(X) = (0^{-\mathbb{N}}10.\xi_{-m}\cdots\xi_{-1}\cdots, g_a(t)) \in \Sigma_2 \times L_s(a,t).$$

Thus $i_0(Z) = s$ and, by step A, $i_k(Z) = s$, for all $k \in I(Z) \cap [0,+\infty]$. On the other hand, $\tilde{G}_{a,t}^{-1}(Z) = P$ and P is a fixed point, hence $-1 \notin I(Z)$, that is, $I(Z) \subseteq [0,+\infty)$. Thus $Z \in \Omega_s(a,t)$ and, by the $\tilde{G}_{a,t}$- invariance of $\Omega_s(a,t)$, we conclude that $X \in \Omega_s(a,t)$.

As $\Omega_s(a,t)$ is a closed set and $H(P,\tilde{G}_{a,t}) \subseteq \overline{W^u(P,\tilde{G}_{a,t})}$, we get

$$H(P,\tilde{G}_{a,t}) \subseteq \Omega_s(a,t).$$

The assertion for $W^s(Q,\tilde{G}_{a,t})$ follows analogously.

Proof of step C. For $X \in W^u(P,\tilde{G}_{a,t}) \cap W^s(Q,\tilde{G}_{a,t})$, we conclude, by step B, that

$$X \in \Omega_s(a,t) \cap \Omega_u(a,t),$$

but $\Omega_s(a,t) \cap \Omega_u(a,t) = \emptyset$, consequently we have $W^s(P,\tilde{G}_{a,t}) \cap W^u(Q,\tilde{G}_{a,t}) = \emptyset$ (no-cycles condition) and **(1)** is proved.

The goal in **(2)** is to prove that any point $X \in W^s(P,\tilde{G}_{a,t}) \cap W^u(Q,\tilde{G}_{a,t})$ is wandering. To see this we divide the proof in four different cases: $I(X) \neq \emptyset$ with $i_0(X) = u$, $I(X) \neq \emptyset$ with $i_0(X) = c$, $I(X) \neq \emptyset$ with $i_0(X) = s$ and $I(X) = \emptyset$.

Case 1. $I(X) \neq \emptyset$ with $i_0(X) = u$

In this case, if there exists a first $k' > 0$ such that $i_{k'}(X) = s$, then, from Lemma 5.3, one has $i_k(X) = s$ for every $k \in I(X) \cap (k',+\infty)$. But this implies that there is a neighborhood $\mathcal{U}$ of $\tilde{G}_{a,t}^{k'-1}(X)$ so that $\tilde{G}_{a,t}^{j}(\mathcal{U}) \cap \mathcal{U} = \emptyset$, for every $j \in \mathbb{N}$, and therefore X is wandering.

Otherwise, that is, if $i_k(X) = u$ for every $k \in I(X) \cap [0, +\infty)$, then, by replacing X by some iterate we can consider

$$X = \big(\cdots \beta_{-1}.0^{\mathbb{N}}, x\big) \in W^s(P, \tilde{G}_{a,t}), \quad \text{with } x \in L_u(a,t),$$

and, from the definition of X and noting that $\tilde{g}_{1,t}([g_a(1/2 - t), 1/2]) \subseteq L_s(a,t)$, we conclude that there is a neighborhood $\mathcal{U}$ of X so that, for every $Y \in U$, one has $i_0(Y) = u$ and $i_k(Y) = s$ for every $k \in I(Y) \cap [1, +\infty)$. Consequently X is wandering.

Case 2. $I(X) \neq \emptyset$ with $i_0(X) = c$

In this case it is clear that X is wandering.

Case 3. $I(x) \neq \emptyset$ with $i_0(X) = s$

As in *case 1*, either X is wandering or, replacing X by some iterate, we can assume that

$$X = \big(0^{-\mathbb{N}}.\beta_0 \cdots, x\big) \in W^u(Q, \tilde{G}_{a,t}), \quad \text{with } x \in L_s(a,t).$$

Since $\tilde{g}_{1,t}^{-1}([0, d_t]) \subset \big[1/2 - t, g_a^n(s_{a,t}^-)\big]$, there is a neighborhood $\mathcal{U}$ of X so that for every $Y \in \mathcal{U}$ it holds $i_0(Y) = s$ and $i_k(Y) = u$ for all $k \in I(Y) \cap (-\infty, -1]$, thus $\tilde{G}_{a,t}^{-k}(\mathcal{U}) \cap \mathcal{U} = \emptyset$, for every $k > 0$, and X is also wandering.

Case 4. $I(X) = \emptyset$

Since $X \in W^s(P, \tilde{G}_{a,t}) \cap W^u(Q, \tilde{G}_{a,t})$, then $I(X) = \emptyset$ implies $X \in \{0\}^{\mathbb{Z}} \times (0, 1/2)$. Assume that $X = (0^{\mathbb{Z}}, x) \in \{0\}^{\mathbb{Z}} \times D_{a,t}$. If $x \in L_u(a,t) \cup L_c(a,t)$, then, as above, there is a neighborhood $\mathcal{U}$ of X such that $i_0(Y) \in \{u, c\}$ and $i_k(Y) = s$ for every $Y \in \mathcal{U}$ and $k \in I(Y) \cap [1, +\infty)$. Therefore $\tilde{G}_{a,t}^j(Y) \notin \mathcal{U}$ for all $j \in \mathbb{N}$ and $Y \in \mathcal{U}$. If $x \in L_s(a,t)$, then, arguing as in *case 1*, we conclude that X is wandering.

The assertion **(3)** and (**4**) follow from the fact that if $X \in W^s(P, \tilde{G}_{a,t}) \cap W^u(P, \tilde{G}_{a,t})$ (respectively $X \in W^s(Q, \tilde{G}_{a,t}) \cap W^u(Q, \tilde{G}_{a,t})$), then $X \in \Lambda_{a,t} \setminus W^s(Q, \tilde{G}_{a,t})$ (respectively $X \in \Lambda_{a,t} \setminus W^u(P, \tilde{G}_{a,t})$).

Proof of step D. Let $X \in \Omega(\tilde{G}_{a,t})' \cap \tilde{\Delta}_{a,t}$. If $I(X)$ is finite, then

$$X \in \Big(W^u(P, \tilde{G}_{a,t}) \cup W^u(Q, \tilde{G}_{a,t})\Big) \cap \Big(W^s(P, \tilde{G}_{a,t}) \cup W^s(Q, \tilde{G}_{a,t})\Big),$$

which implies that $X \in H(P, \tilde{G}_{a,t}) \cup H(Q, \tilde{G}_{a,t})$ or

$$X \in (W^s(P, \tilde{G}_{a,t}) \cap W^u(Q, \tilde{G}_{a,t})) \cup (W^u(P, \tilde{G}_{a,t}) \cap W^s(Q, \tilde{G}_{a,t})).$$

Since $X \in \Omega(\tilde{G}_{a,t})$, by **(1)** and **(2)** of step C , we have $X \in H(P, \tilde{G}_{a,t}) \cup H(Q, \tilde{G}_{a,t})$. Thus, from step B, we conclude that $X \in \Omega_s(a,t) \cup \Omega_u(a,t)$.

In the case $I(X)$ infinite, we claim that

$$i_j(X) = i_k(X), \text{ for every } j,\ k \in I(X).$$

Consequently either $i_j(X) = s$ for all $j \in I(X)$, that is, $X \in \Omega_s(a,t)$, or $i_j(X) = u$ for every $j \in I(X)$ and we have $X \in \Omega_u(a,t)$.

To get a contradiction, assume that $i_j(X) \neq i_{j+1}(X)$, for some $j,\ j+1 \in I(X)$. Replacing X by some iterate of it we can take $j = 0$. From step A, we know that $i_0(X) \in \{u, c\}$ and $i_1(X) \in \{s, c\}$ and all the possibilities imply that $X \notin \Omega(\tilde{G}_{a,t})$, which is a contradiction.

Proof of step E. Take $X \in \Omega(\tilde{G}_{a,t})'$. If $I(X) = \emptyset$, then $X \in \{0\}^{\mathbb{Z}} \times [0, 1/2]$ and, since X is a non-wandering point, we have

$$X \in \{0\}^{\mathbb{Z}} \times \{0, 1/2\} = \{P, Q\} \subset \Omega_s(a,t) \cup \Omega_u(a,t).$$

Otherwise, if $I(X) \neq \emptyset$, then we can assume that $X \in \tilde{\Delta}_{a,t} = \Sigma_2 \times D_{a,t}$. From steps B and D we get

$$H(P, \tilde{G}_{a,t}) \cup H(Q, \tilde{G}_{a,t}) \subseteq \Omega(\tilde{G}_{a,t})',$$

so it suffices to prove $\Omega_u(a,t) \subseteq H(Q, \tilde{G}_{a,t})$ and $\Omega_s(a,t) \subseteq H(P, \tilde{G}_{a,t})$. Let us prove this fact for $\Omega_u(a,t)$; the other assertion follows similarly.

Taking $X \in \Omega_u(a,t)$, we divide the proof in four different cases: the sets

$$I^+(X) := I(X) \cap (0, +\infty) \text{ and } I^-(X) := I(X) \cap (-\infty, 0)$$

are both finite; $I^+(X)$ is finite and $I^-(X)$ is infinite; $I^+(X)$ is infinite and $I^-(X)$ is finite; $I^+(X)$ and $I^-(X)$ are both infinite.

Case 1. If $I^+(X)$ and $I^-(X)$ are finite, then

$$X \in W^s(Q, \tilde{G}_{a,t}) \cap W^u(Q, \tilde{G}_{a,t}) = W^s(Q, \tilde{G}_{a,t}) \pitchfork W^u(Q, \tilde{G}_{a,t}),$$

and we conclude that $X \in H(Q, \tilde{G}_{a,t})$.

Case 2. If $I^+(X)$ is finite, then $X = (\xi, x) = (\cdots \xi_{-1}.\xi_0 \cdots \xi_l 0^{\mathbb{N}}, x) \in W^s(Q, \tilde{G}_{a,t})$. Since $X \in \Lambda_{a,t}$ and $I^-(X)$ is infinite, then the point

$$X_m = (0^{-\mathbb{N}} \xi_{-m} \cdots \xi_{-1}.\xi_0 \cdots \xi_l 0^{\mathbb{N}}) \in W^s(Q, \tilde{G}_{a,t}) \cap W^u(Q, \tilde{G}_{a,t}), \quad \text{for } m \in \mathbb{N} \text{ large},$$

thus $X_m \in H(Q, \tilde{G}_{a,t})$ and $X_m \to X$ as $m \to \infty$. Therefore $X \in H(Q, \tilde{G}_{a,t})$.

Case 3. The set $I^+(X)$ is infinite and $I^-(X)$ is finite.

Since $X \in \Omega_u(a,t)$ and $I^+(X)$ is infinite, from Lemma 5.5 there is a sequence $\{u_j\}_{j \in \mathbb{N}}$ such that

$$X = (\xi, x) = (0^{\mathbb{N}} \xi_{-l} \cdots \xi_{-1}.0^n 1 0^{u_1} 0^n 1 0^{u_2} \cdots 0^n 1 0^{u_j} \cdots) \in W^u(Q, \tilde{G}_{a,t}).$$

Letting $m \in \mathbb{N}$ large, from the definition of X there are a chain $\mathfrak{b} = (u_1, 0) \cdots (u_m, 0)$ and $m' > m$ such that $[x - 1/m', x + 1/m'] \subseteq D^{\mathfrak{b}}_{a,t}$ and

$$g^{-1}_{[\xi_{-l} \cdots \xi_{-1}],t}\left(\left[x - \frac{1}{m'}, x + \frac{1}{m'}\right]\right) \subseteq \left[0, \frac{1}{2}\right].$$

Once, from Lemma 5.5, the restriction of $\tilde{\Gamma}^{(u,0)}_{a,t}$ to $L_u(a,t)$ is uniformly expanding for all $u \in \mathbb{N}$, then there is $m'' \in \mathbb{N}$ such that

$$d_t \in g_{[0^n 1 0^{u_1} \cdots 0^n 1 0^{u_{m''}}],t}\left(\left[x - \frac{1}{m'}, x + \frac{1}{m''}\right]\right)$$

and consequently there is $x_m \in [x - 1/m'', x + 1/m'']$ such that $g_{[0^n 1 0^{u_1} \cdots 0^n 1 0^{u_{m''}}],t}(x_m) = d_t$ and

$$X_m = \left(0^{\mathbb{N}} \xi_{-l} \cdots \xi_{-1}.0^n 1 0^{u_1} \cdots 0^n 1 0^{u_{m''}} 0^{\mathbb{N}}, x_m\right) \in H(Q, \tilde{G}_{a,t}).$$

Since $X_m \to X$ as $m \to +\infty$, one gets $X \in H(Q, \tilde{G}_{a,t})$.

Finally, the case 4, $I^+(X)$ and $I^-(X)$ infinite sets, cames from the cases 2 and 3, and the step E is proved.

Proof of step F. Let us see that $H(P,\tilde{G}_{a,t})$ and $H(Q,\tilde{G}_{a,t})$ are basic sets. We need to prove that $H(P,\tilde{G}_{a,t})$ and $H(Q,\tilde{G}_{a,t})$ are invariant, compact, transitive (they have a dense orbit), isolated, or locally maximal, that is, there are neighborhoods U_1 and U_2 of $H(P,\tilde{G}_{a,t})$ and $H(Q,\tilde{G}_{a,t})$, respectively, such that

$$H(P,\tilde{G}_{a,t}) = \bigcap_{n\in\mathbb{Z}} \tilde{G}_{a,t}(U_1) \text{ and } H(Q,\tilde{G}_{a,t}) = \bigcap_{n\in\mathbb{Z}} \tilde{G}_{a,t}(U_2),$$

and they have a dense subset of periodic orbits.

The invariance, the compactness, the transitivity of the sets and the density of periodic orbits come from the definition of homoclinic classes.

Next we prove that $\Omega_s(a,t) = H(P,\tilde{G}_{a,t})$ is locally maximal. For $\Omega_u(a,t)$, the situation is similar. We claim that

$$\Omega_s(a,t) = \bigcap_{j\in\mathbb{Z}} \tilde{G}^j_{a,t}\left(\Sigma_2 \times \left[s^+_{a,t}, \frac{1}{2}\right]\right).$$

The inclusion $\Omega_s(a,t) \subseteq \bigcap_{j\in\mathbb{Z}} \tilde{G}^j_{a,t}\left(\Sigma_2 \times [s^+_{a,t}, 1/2]\right)$ follows from Lemma 5.3. For the converse take

$$X \in \bigcap_{j\in\mathbb{Z}} \tilde{G}^j_{a,t}\left(\Sigma_2 \times \left[s^+_{a,t}, \frac{1}{2}\right]\right) \subset \Lambda_{a,t}.$$

If $I(X) = \emptyset$ then $X \in 0^{\mathbb{Z}} \times [0,1/2]$ and, since the backward orbit of any point on $0^{\mathbb{Z}} \times [0,1/2)$ meets $\Sigma_2 \times [0,s^+_{a,t})$, we can conclude that $X = (0^{\mathbb{Z}},1/2) = P \in \Omega_s(a,t)$. If $I(X)$ is finite, from Lemma 5.6 and **(1)** of step C, we conclude that

$$X \in H(Q,\tilde{G}_{a,t}) \cup H(P,\tilde{G}_{a,t}) \cup \left(W^s(P,\tilde{G}_{a,t}) \cap W^u(Q,\tilde{G}_{a,t})\right)$$

and, consequently $X \in H(P,\tilde{G}_{a,t}) = \Omega_s(a,t)$ because if $X \in W^u(Q,\tilde{G}_{a,t})$ then its backward orbit would intersect $\Sigma_2 \times [0,s^+_{a,t}]$, which is not possible. Finally, if $I(X)$ is infinite, by definition of X, we have $i_k(X) = s$ for all $k \in I(X)$ and consequently we just need to see that $X \in \Omega(\tilde{G}_{a,t})$. In similar way as above, we can show that there is a sequence of periodic orbits $(X_m)_m$ in $\Omega_s(a,t)$ such that $X_m \to X$, as $m \to \infty$. As $\Omega_s(a,t) \subset \Omega_s(\tilde{G}_{a,t})$, which is a closed set, we have $X \in \Omega(\tilde{G}_{a,t})$ and consequently $\Omega_s(a,t) = \bigcap_{j\in\mathbb{Z}} \tilde{G}^j_{a,t}\left(\Sigma_2 \times [s^+_{a,t}, 1/2]\right)$.

Proof of step G. It is enough to see that $W^s(\Omega_u(a,t), \tilde{G}_{a,t}) \cap W^u(\Omega_s(a,t), \tilde{G}_{a,t}) = \emptyset$, where

$$W^j(\Omega_k(a,t), \tilde{G}_{a,t}) = \bigcup_{x\in\Omega_k(a,t)} W^j(x, \tilde{G}_{a,t}),\ j = s,\ u \text{ and } k = s,\ u.$$

In fact observe that every $Z \in W^s(\Omega_u(a,t), \tilde{G}_{a,t}) \cap W^u(\Omega_s(a,t), \tilde{G}_{a,t})$ belongs to $\Omega(\tilde{G}_{a,t})'$ and arguing as in the proof of **(1)** in step C we get j and k such that $i_k(Z) = s$ and $i_j(Z) = u$. But, by step A, this is impossible.

Now we are in position to finish the proof of the main theorem of this chapter.

Proof of Theorem 5.1. The first item follows by construction of the sequence t_n. Recall that

$$0^{\mathbb{Z}} \times (0, 1/2) \in W^s(P, \tilde{G}_{a,t}) \cap W^u(Q, \tilde{G}_{a,t})$$

and, from de equation $g^n_a(t_n) = 1/2 - t_n$, we have

$$(0^{-\mathbb{N}}.10^{\mathbb{N}}, 1/2) \in W^s(Q, \tilde{G}_{a,t}) \cap W^u(P, \tilde{G}_{a,t}).$$

The second item follows from the steps A-G and from the equations (5.1.3), (5.2.3) and (5.2.4). In fact,

$$\begin{aligned} \lim_{n\to+\infty} \frac{t_n - \mu^\star_{a,t_n}}{t_n - t_{n+1}} &= \lim_{n\to+\infty} \frac{t_n\left(1 - \frac{\mu^\star_{a,t_n}}{t_n}\right)}{t_n\left(1 - \frac{t_{n+1}}{t_n}\right)} \\ &= \frac{1 - 2e^{-\frac{a}{2}}}{1 - e^{-\frac{a}{2}}} = \frac{e^{\frac{a}{2}} - 2}{e^{\frac{a}{2}} - 1} > 0,\ \forall a > \log 4. \end{aligned}$$

The last item follows from $s_{a,t_n} = L_s(a, \mu^\star_{a,t_n}) \cap L_u(a, \mu^\star_{a,t_n})$, Lemmas 5.3, 5.4, 5.5, and that s_{a,t_n} is a saddle-node of $\tilde{\Gamma}^{(1,0)}_{a,\mu^\star_{a,t_n}}$. □

5.4 Hyperbolicity of the model family of heterodimensional cycles

The proof of Theorem 5.1 can be addapted to the one-parameter family $f_{a,t}$, where $t > 0$ small and $a > \log 4$, defined in Section 2.4. In fact we have the following result:

Theorem 5.7. *Let $a > \log 4$. Then there are $t_0(a) > 0$, a sequence $t_n(a) \in (0, t_0(a)]$ converging to zero as $n \to \infty$, and $\mu^{\star}_{a,t_n} \in (t_{n+1}, t_n)$ such that:*

1. *for every parameter $t \in \left(\mu^{\star}_{a,t_n}, t_n\right)$, the resulting nonwandering set of $f_{a,t}$ is hyperbolic and equal disjoint union of the (non-trivial) homoclinic classes os P and Q.*
2. *$f_{a,\mu^{\star}_{a,t_n}}$ has a saddle-node S_{a,t_n} such that the intersection of the homoclinic classes of P and Q is exactly the orbit of S_{a,t_n}.*
3. *Moreover, $\displaystyle\lim_{a\to+\infty}\left(\lim_{n\to\infty}\frac{t_n-\mu^{\star}_{a,t_n}}{t_n-t_{n+1}}\right)=1.$*

The idea is to reduce the study of the dynamics in a neighborhood of the cycle to one dimensional dynamics. In this way, we get a system $\mathfrak{F}_{a,t}$ of iterated functions with the same hyperbolicity-like properties that $\tilde{\mathfrak{G}}_{a,t}$. In fact, they are equal. The theorem follows, using the existence of the filtration and the geometry of the cycle, by translating the hyperbolicity of the system $\mathfrak{F}_{a,t}$ to the diffeomorphisms $f_{a,t}$, for all $t \in (\mu^{\star}_{a,t_n}, t_n)$. Since the proof of this result follows using similar arguments to the ones used in the proof of Theorem 5.1 with the natural adaptations, we will omit the proof.

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