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Optimal Control

A Numerical Algorithm for Optimal Control Problems with a Viscous Point Vortex

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Abstract. The dynamics of passive tracers in flows dominated by perfect or viscous point vortices is a broad area of research that continues to attract the attention of numerous studies. Recently, there has been a particular interest in the application of control theory to these issues. Viscous point vortices are singular solutions of the two-dimensional incompressible Navier-Stokes equations in which the vorticity is concentrated at a finite number of points in the flow domain, each of which carries a certain amount of time-invariant circulation. By definition, a passive tracer is a point vortex with zero circulation. This paper describes some numerical investigations of passive tracers performed by viscous point vortices to find the energy-optimal displacement of a passive particle. The numerical results show the existence of near/quasi-optimal controls.

Keywords: Viscous point vortex · Passive tracer · Control problem · Numerical optimization

1 Introduction

This article is concerned with the dynamics of a passive particle carried away by a two-dimensional viscous point vortex flow. A point vortex is a mathematical model used to describe the dynamics of vortex-dominated flows [1], based on a low-dimensional description of the flow features [2]. A passive particle is small enough not to perturb the velocity field, but also large enough not to perform a Brownian motion. Particles of this type are, for example, the tracers used to visualize flows in fluid mechanics [3].

We want to move a passive particle from an initial starting point to a final destination point, both known a priori, in a given (finite) time. The flow is generated by N point vortices, point vortices, as in the problem of fish-like locomotion [4,5]. Here, the vortex dynamics is governed by N viscous point vortices and the control arises from the possibility of propulsion in any direction of the two-dimensional plane. We want to minimize the energy spent on propulsion. This kind of problem belongs to a set of open problems mentioned in Protas [6].

The solution of the proposed problem was achieved by direct and indirect control methods. The indirect method uses Pontryagin's maximum principle to derive optimal

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conditions, where it is necessary to maximize the Hamiltonian, which can be achieved by collocation or shooting methods [7]. For energy costs, the candidates for minimizers are given by the normal extremes of Pontryagin's maximum principle. The rewriting of Pontryagin's maximum principle yields a set of nonlinear equations to be solved, called the shooting equations. In [8], the shooting equations are introduced and numerical calculations are presented in the cases of $N = 1, 2, 3$ and 4 perfect point vortices.

In the solution by direct approach, the problem is discretized with respect to time to obtain a nonlinear programming problem (NLP) that can be solved by an optimization method. In [9], the resulting optimization problems were solved numerically. The numerical results show the existence of near/quasi-optimal controls in the case of vortex dynamics resulting from the interaction of $N = 1, 2, 3, 4$ point vortices. In [10] it was shown that the more control points there are, the less energy is required to make the displacement. In general, the energy required for the motion increases with the number of point vortices. The same is true for the computation time required for the optimization method. It increases with the number of control points as well as with the number of vortices.

All previous solutions of this problem by direct or indirect methods did not take into account the dynamic viscosity, i.e., the displacement of particles interacting with N point vortices was controlled in a two-dimensional fluid and viscous diffusion was neglected. In the present study, the passive particle moves in a two-dimensional viscous flow whose dynamics is given by N point vortices in each time interval. We consider different cases in which, in addition to the dynamics induced by a different number of vortices, different dynamic viscosity values are also considered. Four different problems are considered. Each corresponds to a different value of N, ranging from 1 to 4. The nonlinear programming problem corresponding to each of these cases is formulated and solved numerically using Matlab Optimization Toolbox [11].

Section 2 presents the equations for the motion of passive particles carried away by N viscous point vortices in the infinite real plane and the formulation of the control problems. Section 3 is devoted to the numerical solutions for these control problems. This paper ends in Sect. 4 with the presentation of some concluding considerations.

2 The Viscous Point Vortex Control Problem

Perfect point vortices are singular solutions of the two-dimensional incompressible Euler equation [1,3,10]. These solutions correspond to the limiting case in which the vortices are concentrated on a finite number of spatial points, each of which has a prescribed circulation. Using the complex variable z_α to denote the position of the vortex α with circulation k_α ($\alpha = 1, 2, \ldots, N$), the differential equations governing a set of N (non-viscous) vortices are as follows.

$$\frac{dz_\alpha^*}{dt} = \frac{1}{2\pi i} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \frac{k_\beta}{z_\alpha - z_\beta} \quad (\alpha = 1, 2, \ldots, N)\,, \tag{1}$$

with the respective initial conditions, where $z_\alpha = x_\alpha + y_\alpha\, i$ $\left(i^2 = -1\right)$ and z_α^* is its complex conjugate.

A passive particle is, by definition, a point vortex with zero circulation. Therefore, the dynamics of a system with P passive particles advected by a set of N point vortices is given by Eq. (1) together with the equations for the passive particles

$$\frac{dz_\alpha^*}{dt} = \frac{1}{2\pi i} \sum_{\beta=1}^{N} \frac{k_\beta}{z_\alpha - z_\beta} \quad (\alpha = N+1, N+2, \ldots, N+P)\,, \tag{2}$$

with the respective initial conditions.

Given now that the vortices and passive particles are both moving in a viscous fluid with a kinematic viscosity of $\nu > 0$, the system with P passive particles advected by N viscous point vortices is then given by

$$\frac{dz_\alpha^*}{dt} = \frac{1}{2\pi i} \sum_{\substack{\beta = 1 \\ \beta \neq \alpha}}^{N} \frac{k_\beta}{z_\alpha - z_\beta} \left[1 - \exp\left(-\frac{|z_\alpha - z_\beta|^2}{4\nu t}\right)\right],\ \alpha = 1, 2, \ldots, N \tag{3}$$

and

$$\frac{dz_\alpha^*}{dt} = \frac{1}{2\pi i} \sum_{\beta=1}^{N} \frac{k_\beta}{z_\alpha - z_\beta} \left[1 - \exp\left(-\frac{|z_\alpha - z_\beta|^2}{4\nu t}\right)\right] + \mathbf{U}(t)\,, \quad \alpha = N+1, N+2, \ldots, N+P, \tag{4}$$

with the respective initial conditions (see, for instance, [12]). In Eq. 4, $\mathbf{U}(t)$ is the control function.

In this work, we consider a single passive particle ($P = 1$) advected by $N = 1, 2, 3, 4$ viscous point vortices. The control problem is to move this tracer between two given points in a given fixed time (T) while consuming as little energy as possible.

2.1 Solving the Control Problem by Direct Approach

As mentioned above, the numerical approach used to solve the control problem presented above is based on a direct approach. It consists in discretizing the problem and solving it using an optimization method. Thus, the control function $\mathbf{U}(\cdot)$ is replaced by n control variables $u_0,\ u_1, \cdots,\ u_{n-1}$. Numerical calculations were performed in Matlab using the nonlinear programming solver `fmincon`. This solver provides some constrained optimization algorithms, such as the interior point or the active set (see [11]). We start in the Subsect. 2.2, with the solution of this problem for a single passive particle in a viscous single-vortex flow and then, in Sect. 2.3, the cases with up to four vortices are treated. We would like to point out that the cases $N = 2$ and $N = 3$ correspond to integrable point vortex dynamics, while $N = 4$ (or higher) corresponds (in general) to chaotic point vortex dynamics [3, 13].

2.2 Flow Created by One Single Vortex

From a practical point of view, the control problem introduced above can be illustrated as follows for the case of a single passive particle ($P = 1$) moving thanks to the presence of a single vortex ($N = 1$):

$$
\text{Problem } (\mathscr{P}) : \quad
\begin{aligned}
&\textstyle\int_0^T |u(t)|^2 \, dt \longrightarrow \min \,, \\
&\dot{z}^\star = \frac{1}{2\pi i} \frac{k}{z} \left[1 - \exp\left(- \frac{|z|^2}{4\nu t} \right) \right] + u \,, \\
&z(0) = z_0 \,, \\
&z(T) = z_f \,, \\
&|u| \leq u_{\max} \,,
\end{aligned}
$$

with $u \in \mathbb{C}$, and $z_0 \,, z_f \in \mathbb{C}$, $T > 0$ and $u_{\max} > 0$ given.

For example, in this optimization problem, the objective function (cost function) represents the energy expended by the controller $u(\cdot)$ to move the passive particle from the starting point z_0 to the end point z_f. The first constraint corresponds to the equation of state that specifies the position z of the particle as a function of time. The control function u is introduced into this equation to move the particle from z_0 to z_f at a fixed time value $T > 0$.. The points z_0 and z_f are previously defined, as is the time T available to reach the destination point z_f.. In addition, we specify in the fourth constraint that the absolute value of the control is not greater than a given value $u_{\max}$.

To solve this problem, we proceed to the discretization of the control function following [10]. We replace $u(\cdot)$ by n (discrete) variables defined as follows ($t_0 = 0$, $t_n = T$):

$$
\begin{aligned}
u(t) &= u_0 && \text{if } t_0 \leq t < t_1 \,, \\
u(t) &= u_1 && \text{if } t_1 \leq t < t_2 \,, \\
u(t) &= u_2 && \text{if } t_2 \leq t < t_3 \,, \\
& && \vdots \\
u(t) &= u_{n-1} && \text{if } t_{n-1} \leq t \leq t_n \,.
\end{aligned}
$$

Thus, each variable u_i ($i = 1, 2, \cdots, n$) corresponds to a constant value of the control exercised in the sub-interval $[t_{i-1} \,, \, t_i)$. All these sub-intervals have amplitudes equal to $\Delta t = (t_n - t_0)/n$. The discretization of the objective function by the rectangle method lead to the approximation

$$
\int_0^T |u(t)|^2 \, dt \approx \Delta t \sum_{k=0}^{n-1} |u_k|^2 \equiv f_n \,. \tag{5}
$$

As in the study carried out in [10], increasing the number n of control variables (corresponds to reduce Δt) leads to a decrease in the discretized energy to perform the desired movement. The control problem ($\mathscr{P}$) is then replaced by its discretized version:

Discretized Problem ($\mathscr{DP}$)
$\Delta t \sum_{k=0}^{n-1} |u_k|^2 \longrightarrow \min\,,$
$\dot{z}^\star = \frac{1}{2\pi i} \times \frac{k}{z} \times \left[1 - \exp\left(-\frac{|z|^2}{4\nu t}\right)\right] + u_0,\;\; z(0) = z_0,\;\; |u_0| \le u_{\max},\;\; t_0 \le t < t_1\,,$
$\dot{z}^\star = \frac{1}{2\pi i} \times \frac{k}{z} \times \left[1 - \exp\left(-\frac{|z|^2}{4\nu t}\right)\right] + u_1,\;\; z(t_1) = z_{t_1},\;\; |u_2| \le u_{\max},\;\; t_1 \le t < t_2\,,$
$\vdots$
$\dot{z}^\star = \frac{1}{2\pi i} \times \frac{k}{z} \times \left[1 - \exp\left(-\frac{|z|^2}{4\nu t}\right)\right] + u_{n-1},\;\; z(t_{n-1}) = z_{t_{n-1}},\;\; |u_{n-1}| \le u_{\max},$
$t_{n-1} \le t < t_n\,, z(t_n) = z_f\,.$

In the numerical results obtained with the optimization problem $\mathscr{DP}_n$, where n is the number of control variables, (we fix $n = 4$, which corresponds to four control variables), a single vortex with circulation $k = 10$ located at the origin, a vector $u \in \mathbb{C}^n$ is sought that drives the passive particle from $z_0 = -1 - i$ to $z_f = 2 + 2\,i$ in $T = 10$ (natural) time units, minimizing the objective function defined by the r.h.s. of (5). This optimization problem is solved numerically using the interior point optimization algorithm [14], which is included in the `fmincon` Matlab solver. The results obtained with a single vortex are shown in Table 1.

2.3 Flow Created by Several Vortices

In this subsection, we address the problem of a single passive particle ($P = 1$) displaced by multiple vortices (N). As before, we want to find $u = [u_0, u_1, \ldots, u_{n-1}] \in \mathbb{C}^n$ that drags the particle from $z_0 = -1 - i$ to the final destination $z_f = 2 + 2\,i$. We also consider the time of displacement $T = 10$, and equal circulation for all the vortices $k_i = 10$, for $i = 2, 3, 4$. To solve the different optimization problems, we used the `fmincon` Matlab solver. This solver offers different optimization methods, e.g. the Interior Points, Sequential Quadratic Problem (SQP), and Trust Region (see [11]). We tried all these methods and came to the conclusion that the best results were given by the Interior Points optimization method.

Two Vortices ($N = 2$) **and One Particle** ($P = 1$). In the two vortices and one particle problem, the vortices positions are given by [15]:

$$\begin{cases} \dot{z}_1^*(t) = \frac{k_2}{2\pi i(z_1 - z_2)} \left[1 - \exp\left(-\frac{|z_1 - z_2|^2}{4\nu t}\right)\right] \\ \dot{z}_2^*(t) = \frac{k_1}{2\pi i(z_2 - z_1)} \left[1 - \exp\left(-\frac{|z_2 - z_1|^2}{4\nu t}\right)\right] \end{cases} \tag{6}$$

The initial vortex positions are $z_1(0) = 0.5 + 0.5\,i$, and $z_2(0) = 1.5 - 0.5\,i$. The passive particle position is given by the equation

$$\dot{z}^* = \frac{1}{2\pi i} \sum_{i=1}^{2} \frac{k_i}{z - z_i} \left[1 - \exp\left(-\frac{|z - z_i|^2}{4\nu t}\right)\right] + u \tag{7}$$

with the given initial condition $z(0) = z_0$. The results obtained with two vortex are presented in Table 1 and Fig. 1b.

Three Vortices ($N = 3$) **and One Particle** ($P = 1$). In the problem with three vortices ($N = 3$) and one particle ($P = 1$), the vortices equations are

$$\begin{cases} \dot{z}_1^* = \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j \neq 1}}^{3} \frac{k_j}{z_1 - z_j} \left[1 - \exp\left(-\frac{|z_1 - z_j|^2}{4\nu t}\right)\right] \\ \dot{z}_2^* = \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j \neq 2}}^{3} \frac{k_j}{z_2 - z_j} \left[1 - \exp\left(-\frac{|z_2 - z_j|^2}{4\nu t}\right)\right] \\ \dot{z}_3^* = \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j \neq 3}}^{3} \frac{k_j}{z_3 - z_j} \left[1 - \exp\left(-\frac{|z_3 - z_j|^2}{4\nu t}\right)\right] \end{cases} \tag{8}$$

with the given initial boundary $z_1(0) = 0.5+0.5i$, $z_2(0) = 1.5-0.5i$, and $z_3(0) = 1+i$. We consider also the same circulation for all the vortices $k_1 = k_2 = k_3 = 10$. The passive particle equation is

$$\dot{z}^* = \frac{1}{2\pi i} \sum_{i=1}^{3} \frac{k_i}{z - z_i} \left[1 - \exp\left(-\frac{|z - z_i|^2}{4\nu t}\right)\right] + u\,, \tag{9}$$

with the initial condition $z(0) = z_0 = -1 - i$. The results obtained with three vortex, for different values of the kinetic viscosity ν, are presented in Table 1 and Fig. 1 c).

Four Vortices ($N = 4$) **and One Particle** ($P = 1$). We address now the case of four vortices ($N = 4$) and one particle ($P = 1$). This is an interesting case, because the dynamics of the vortices is nonintegrable [3]. For $N = 4$, the vortices equations are

$$\begin{cases} \dot{z}_1^* = \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j \neq 1}}^{4} \frac{k_j}{z_1 - z_j} \left[1 - \exp\left(-\frac{|z_1 - z_j|^2}{4\nu t}\right)\right] \\ \dot{z}_2^* = \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j \neq 2}}^{4} \frac{k_j}{z_2 - z_j} \left[1 - \exp\left(-\frac{|z_2 - z_j|^2}{4\nu t}\right)\right] \\ \dot{z}_3^* = \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j \neq 3}}^{4} \frac{k_j}{z_3 - z_j} \left[1 - \exp\left(-\frac{|z_3 - z_j|^2}{4\nu t}\right)\right] \\ \dot{z}_4^* = \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j \neq 4}}^{4} \frac{k_j}{z_4 - z_j} \left[1 - \exp\left(-\frac{|z_4 - z_j|^2}{4\nu t}\right)\right] \end{cases} \tag{10}$$

with the respective initial positions $z_1(0) = 0.5 + 0.5\,i$, $z_2(0) = 1.5 - 0.5\,i$, $z_3(0) = 1 + i$ and $z_4(0) = -1 - 2\,i$. The passive particle equation is

$$\dot{z}^* = \frac{1}{2\pi i} \sum_{i=1}^{4} \frac{k_i}{z - z_i} \left[1 - \exp\left(-\frac{|z - z_i|^2}{4\nu t}\right)\right] + u\,, \tag{11}$$

with the initial condition $z(0) = -1 - i$. The results obtained with four vortex, for different values of the kinetic viscosity ν, are also presented in Table 1 and Fig. 1d.

3 Results

Table 1 presents the values of the four optimal or quasi-optimal control variables u_i, $i = 0, 1, 2, 3$, and the corresponding value of the objective function f_4, for different viscosity values: $\nu = 0.001, 0.01, 0.1$ and 1.

For $N = 1$ and $N = 2$, some regularity is observed in the decrease of f_4 with the reduction of viscosity. But this regularity disappears for $N = 3$ and $N = 4$.

Table 1. Optimal controls and corresponding energy values, for N vortices.

ν	u_0	u_1	u_2	u_3	f_4
			$N = 1$		
0.001	$-0.0796 - 0.2506i$	$0.1029 - 0.1326i$	$0.1087 - 0.0391i$	$0.0770 + 0.0017i$	0.29
0.01	$-0.0901 - 0.2352i$	$0.1039 - 0.1495i$	$0.1053 - 0.0539i$	$0.0825 + 0.0066i$	0.29
0.1	$-0.0669 - 0.2350i$	$0.1117 - 0.1078i$	$0.1153 - 0.0145i$	$0.0913 + 0.0179i$	0.26
1	$0.2997 + 0.1488i$	$0.2306 + 0.2497i$	$0.1801 + 0.2820i$	$0.1391 + 0.3118i$	1.14
			$N = 2$		
1	$0.2946 + 0.0946i$	$0.0647 - 0.3738i$	$-0.1183 - 0.0510i$	$0.5353 + 0.7265i$	2.67
2	$0.5490 + 0.4455i$	$0.1178 - 0.4258i$	$0.0771 - 0.2245i$	$0.5642 + 0.7752i$	4.18
3	$-1.0468 - 1.4725i$	$-0.6777 - 0.3393i$	$-0.4984 - 0.5882i$	$-1.1815 - 0.5556i$	15.35
4	$0.9622 + 1.9643i$	$1.7246 + 0.9189i$	$1.9327 + 0.2604i$	$1.9519 - 0.2125i$	40.65
			$N = 3$		
0.001	$0.0829 + 0.1476i$	$-0.1866 + 0.0010i$	$0.3220 + 0.4706i$	$-0.3411 - 1.9162i$	10.44
0.01	$-0.2919 - 0.4427i$	$-0.6357 - 0.0393i$	$-0.2269 + 1.3287i$	$-0.1039 - 1.2736i$	10.34
0.1	$1.4200 + 0.6117i$	$-0.3309 - 0.0875i$	$-0.2048 + 0.1628i$	$0.2955 + 0.3992i$	7.06
1	$0.6659 + 0.8229i$	$0.8213 + 0.1155i$	$0.7723 - 0.2538i$	$0.6486 - 0.4925i$	7.83
			$N = 4$		
0.001	$-1.5416 - 3.1687i$	$-1.8420 - 1.2138i$	$-1.0665 - 2.1639i$	$-3.9888 - 3.9898i$	137.33
0.01	$-0.7574 + 0.9901i$	$0.8134 + 0.9661i$	$-1.4311 + 1.6801i$	$-1.6624 + 2.6356i$	44.32
0.1	$-1.1894 - 2.6479i$	$-1.5231 - 0.8714i$	$-1.2413 - 1.6312i$	$-3.9943 - 2.0793i$	89.96
1	$-0.0908 + 0.7080i$	$0.4214 + 0.6825i$	$0.7840 + 0.4060i$	$0.9310 + 0.0647i$	7.01

Figure 1, showing the trajectories corresponding to each optimal control shown in Table 1, shows that these trajectories become more complex as the number of vortices increases. In most cases where $N > 1$, the displacement of the passive tracer between the start and target points is not straight, but occurs in epicycles to take advantage of the displacement induced by the vortices and move without expending energy.

It has also been shown that higher values of viscosity result in straighter trajectories between the start and target points. This is a consequence of the reduction in velocity with increasing viscosity.

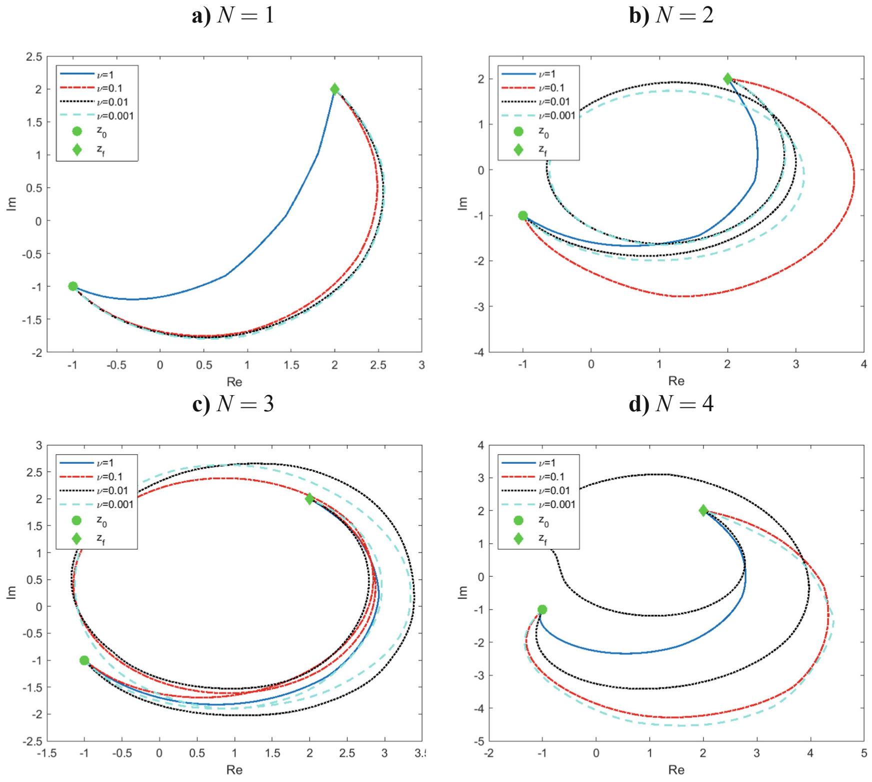

Fig. 1. Optimal trajectories.

4 Conclusion

This work presents the formulation of a set of control problems related to the advection of one passive tracer with $N = 1, 2, 3, 4$ viscous point vortices in the unbounded plane. These control problems arise from the need to move the particle between two points within a certain period of time.

The time span available for the motion is discretized into four subintervals, where the control variable is assumed to be constant, leading to a nonlinear programming problem that is solved numerically. It was found that all problems presented here exhibit near/quasi-optimal control, regardless of the number of vortices for the different values of kinetic viscosity. In general, the energy required for displacement increases with the number of vortex points. The trajectories are straighter for higher values of viscosity.

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