Chapter 2
Anosov and Circle Diffeomorphisms

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Abstract We present an infinite dimensional space of C^{1+} smooth conjugacy classes of circle diffeomorphisms that are C^{1+} fixed points of renormalization. We exhibit a one-to-one correspondence between these C^{1+} fixed points of renormalization and C^{1+} conjugacy classes of Anosov diffeomorphisms.

2.1 Introduction

The link between Anosov diffeomorphisms and diffeomorphisms of the circle is due to D. Sullivan and E. Ghys through the observation that the holonomies of Anosov diffeomorphisms give rise to C^{1+} circle diffeomorphisms that are C^{1+} fixed points of renormalization (see also [1]). A. Pinto and D. Rand [21] proved that this observation gives an one-to-one correspondence between the corresponding smooth

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conjugacy classes. A key object in this link is the smooth horocycle equipped with a hyperbolic Markov map.

2.2 Circle Difeomorphisms

Fix a natural number $a \in \mathbb{N}$ and let $\mathbb{S}$ be a *counterclockwise oriented circle* homeomorphic to the circle $\mathbb{S}^1 = \mathbb{R}/(1+\gamma)\mathbb{Z}$, where $\gamma = (-a+\sqrt{a^2+4})/2 = 1/(a+1/(a+\cdots))$. We note that if $a = 1$ then γ is the inverse of the golden number $(1+\sqrt{5})/2$. A key feature of γ is that it satisfies the relation $a\gamma + \gamma^2 = 1$.

An *arc* in $\mathbb{S}$ is the image of a non trivial interval I in $\mathbb{R}$ by a homeomorphism $\alpha : I \to \mathbb{S}$. If I is closed (resp. open) we say that $\alpha(I)$ is a *closed* (resp. *open*) *arc* in $\mathbb{S}$. We denote by (a,b) (resp. $[a,b]$) the positively oriented open (resp. closed) arc in $\mathbb{S}$ starting at the point $a \in \mathbb{S}$ and ending at the point $b \in \mathbb{S}$. A C^{1+} atlas $\mathscr{A}$ of $\mathbb{S}$ is a set of charts such that (i) every small arc of $\mathbb{S}$ is contained in the domain of some chart in $\mathscr{A}$, and (ii) the overlap maps are $C^{1+\alpha}$ compatible, for some $\alpha > 0$.

A C^{1+} *circle diffeomorphism* is a triple $(g, \mathbb{S}, \mathscr{A})$ where $g : \mathbb{S} \to \mathbb{S}$ is a $C^{1+\alpha}$ diffeomorphism, with respect to the $C^{1+\alpha}$ atlas $\mathscr{A}$, for some $\alpha > 0$, and g is quasi-symmetric conjugate to the rigid rotation $r_\gamma : \mathbb{S}^1 \to \mathbb{S}^1$, with rotation number equal to $\gamma/(1+\gamma)$. We denote by $\mathscr{F}$ the set of all C^{1+} circle diffeomorphisms $(g, \mathbb{S}, \mathscr{A})$, with respect to a C^{1+} atlas $\mathscr{A}$ in $\mathbb{S}$.

In order to simplify the notation, we will denote the C^{1+} circle diffeomorphism $(g, \mathbb{S}, \mathscr{A})$ only by g.

2.2.1 The Horocycle and Renormalization

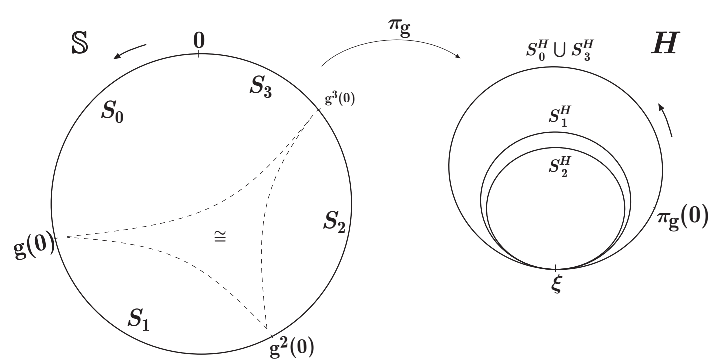

Fig. 2.1 The horocycle H for the case $a = 2$. The junction ξ of the horocycle is equal to $\xi = \pi_g(g(0)) = \pi_g(g^2(0)) = \pi_g(g^3(0))$.

Let us mark a point in $\mathbb{S}$ that we will denote by $0 \in \mathbb{S}$, from now on. Let $S_0 = [0, g(0)]$ be the oriented closed arc in $\mathbb{S}$, with endpoints 0 and $g(0)$. For $k = 0, \ldots, a$ let $S_k = \left[g^k(0), g^{k+1}(0)\right]$ be the oriented closed arc in $\mathbb{S}$, with endpoints $g^k(0)$ and $g^{k+1}(0)$ and such that $S_k \cap S_{k-1} = \{g^k(0)\}$. Let $S_{a+1} = \left[g^{a+1}(0), 0\right]$ be the oriented closed arc in $\mathbb{S}$, with endpoints $g^{a+1}(0)$ and 0. We introduce an *equivalence relation* $\sim$ in $\mathbb{S}$ by identifying the $a+1$ points $g(0), \ldots, g^{a+1}(0)$ and form the oriented topological space $H(\mathbb{S}, g) = \mathbb{S}/\sim$. We call this oriented topological space the *horocycle* (see Figure 2.1) and we denote it by $H = H(\mathbb{S}, g)$. We consider the quotient topology in H. Let $\pi_g : \mathbb{S} \to H$ be the natural projection. The point $\xi = \pi_g(g(0)) = \cdots = \pi_g(g^{a+1}(0)) \in H$ is called the *junction* of the horocycle H. For $k = 0, \ldots, a+1$, let $S_k^H = S_k^H(\mathbb{S}, g) \subset H$ be the projection by π_g of the closed arc S_k. A *parametrization* in H is the image of a non trivial interval I in $\mathbb{R}$ by a homeomorphism $\alpha : I \to H$. If I is closed (resp. open) we say that $\alpha(I)$ is a *closed* (resp. *open*) *arc* in H. A *chart* in H is the inverse of a parametrization. A *topological atlas* $\mathscr{B}$ on the horocycle H is a set of charts $\{(j, J)\}$, on the horocycle, with the property that every small arc is contained in the domain of a chart in $\mathscr{B}$, i.e. for any open arc K in H and any $x \in K$ there exists a chart $\{(j, J)\} \in \mathscr{B}$ such that $J \cap K$ is a non trivial open arc in H and $x \in J \cap K$. A C^{1+} *atlas* $\mathscr{B}$ *in* H is a topological atlas $\mathscr{B}$ such that the overlap maps are $C^{1+\alpha}$ and have $C^{1+\alpha}$ uniformly bounded norms, for some $\alpha > 0$.

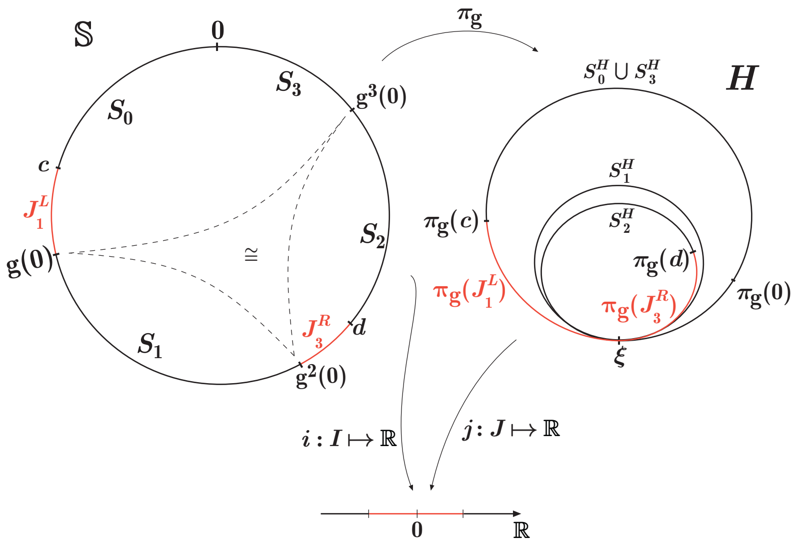

Fig. 2.2 The chart $j : J \to \mathbb{R}$ in case (ii).

Let $\mathscr{A}$ be a C^{1+} atlas in $\mathbb{S}$ for which g is C^{1+}. We are going to construct a C^{1+} atlas $\mathscr{A}^H$ in the horocycle that is the *extended pushforward* $\mathscr{A}^H = (\pi_g)_* \mathscr{A}$ of the atlas $\mathscr{A}$ in $\mathbb{S}$.

If $x \in H \setminus \{\xi\}$ then there exists a sufficiently small open arc J in H, containing x, such that $\pi_g^{-1}(J)$ is contained in the domain of some chart (I,i) in $\mathscr{A}$. In this case, we define $(J, i \circ \pi_g^{-1})$ as a chart in $\mathscr{A}^H$. If $x = \xi$ and J is a small arc containing ξ, then either (i) $\pi_g^{-1}(J)$ is an arc in $\mathbb{S}$ or (ii) $\pi_g^{-1}(J)$ is a disconnected set that consists of a union of two connected components. In case (i), $\pi_g^{-1}(J)$ is connected and we define $\left(J, i \circ \pi_g^{-1}\right)$ as a chart in $\mathscr{A}^H$. In case (ii), $\pi_g^{-1}(J)$ is a disconnected set that is the union of two connected arcs J_k^L and J_l^R of the form $J_l^R = [g^l(0), d)$ and $J_k^L = (c, g^k(0)]$, respectively, for some $k, l \in \{0, \ldots, a+1\}$ with $k \neq l$ (see Figure 2.2). Let $(I,i) \in \mathscr{A}$ be a chart such that $I \supset (c,d)$. We define $j : J \to \mathbb{R}$ as follows,

$$j(x) = \begin{cases} i \circ \pi_g^{-1}(x), & \text{if } x \in \pi_g([g^l(0), d)) \\ i \circ g^{l-k} \circ \pi_g^{-1}(x), & \text{if } x \in \pi_g((c, g^k(0)]) \end{cases} .$$

We call the atlas determined by these charts, the *extended pushforward atlas of* $\mathscr{A}$ and, by abuse of notation, we will denote it by $\mathscr{A}^H = (\pi_g)_* \mathscr{A}$.

Let $g = (g, \mathbb{S}, \mathscr{A})$ be a C^{1+} circle diffeomorphism with respect to a C^{1+} atlas $\mathscr{A}$ in $\mathbb{S}$. Let $R\mathbb{S} = S_0^H \cup S_{a+1}^H$ be projection by π_g of the oriented close arc $S_0 \cup S_{a+1}$ of $\mathbb{S}$. Let $i_{R\mathbb{S}} : S_0 \cup S_{a+1} \subset \mathbb{S} \to R\mathbb{S}$ be the natural inclusion and let $R\mathscr{A}$ be the restriction, $\mathscr{A}|R\mathbb{S}$, of the C^{1+} atlas $\mathscr{A}$ to $R\mathbb{S}$. The *renormalization of* $g = (g, \mathbb{S}, \mathscr{A})$ is the triple $(Rg, R\mathbb{S}, R\mathscr{A})$, where $Rg : R\mathbb{S} \to R\mathbb{S}$ is the map given by (see Figure 2.3),

$$\begin{cases} i_{R\mathbb{S}} \circ g^{a+1} = Rg \circ i_{R\mathbb{S}}, \; for\ Rg|i_{R\mathbb{S}}(S_0), \\ i_{R\mathbb{S}} \circ g = Rg \circ i_{R\mathbb{S}}, \quad for\ Rg|i_{R\mathbb{S}}(S_{a+1}) \end{cases}$$

For simplicity of notation, we will denote the renormalization of a C^{1+} circle diffeomorphism g, $(Rg, R\mathbb{S}, R\mathscr{A})$, only by Rg.

We recall that $\mathscr{F}$ denotes the set of all C^{1+} circle diffeomorphisms $(g, \mathbb{S}, \mathscr{A})$ with respect to a C^{1+} atlas $\mathscr{A}$ in $\mathbb{S}$.

Lemma 2.1. *The renormalization Rg of a C^{1+} circle diffeomorphism $g \in \mathscr{F}$ is a C^{1+} circle diffeomorphism, i.e. the map $R : \mathscr{F} \to \mathscr{F}$ given by $R(g) = Rg$ is well defined. In particular, the renormalization Rr_γ of the rigid rotation is the rigid rotation r_γ.*

The proof of Lemma 2.1 is in [21].

The marked point $0 \in \mathbb{S}$ determines a marked point 0 in the circle $R\mathbb{S}$. Since Rg is homeomorphic to a rigid rotation, there exists $h : \mathbb{S} \to R\mathbb{S}$, with $h(0) = 0$, such that h conjugates g and Rg.

Definition 2.1. If $h : \mathbb{S} \to R\mathbb{S}$ is C^{1+}, we call g a C^{1+} *fixed point of renormalization*. We will denote by $\mathscr{R}$ the set of all C^{1+} circle diffeomorphisms $g \in \mathscr{F}$ that are C^{1+} fixed points of renormalization.

We note that the rigid rotation r_γ, with respect to the atlas $\mathscr{A}_{iso}$, is an affine fixed point of renormalization. Hence, $r_\gamma \in \mathscr{R}$.

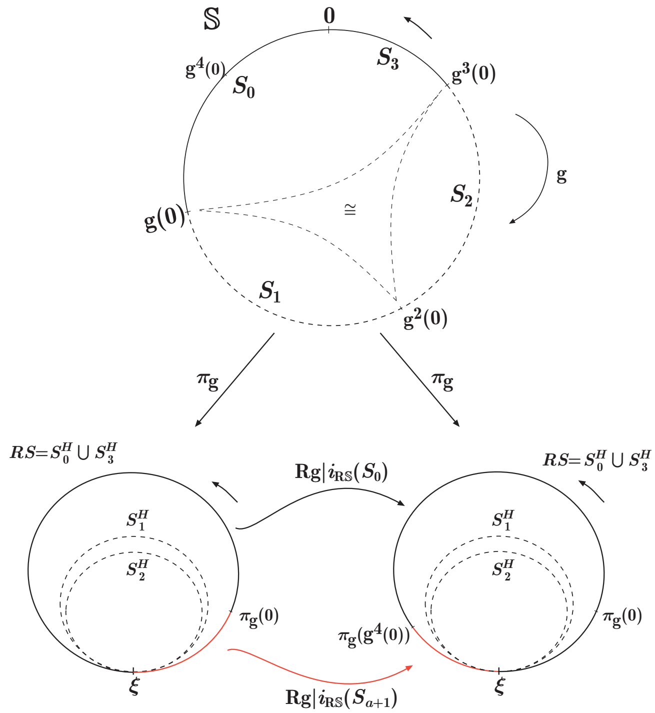

Fig. 2.3 The renormalization $Rg = (Rg, R\mathbb{S}, R\mathscr{A})$ of a C^{1+} circle diffeomophism $g = (g, \mathbb{S}, \mathscr{A})$.

2.2.2 Markov Maps

Let $g = (g, \mathbb{S}, \mathscr{A})$ be a C^{1+} circle diffeomorphism, with respect to a C^{1+} atlas $\mathscr{A}$, and $Rg = (Rg, R\mathbb{S}, R\mathscr{A})$ its renormalization. Let $H = H(\mathbb{S}, g)$ and $RH = H(R\mathbb{S}, Rg)$ be the horocycles determined by the C^{1+} circle diffeomorphisms g and Rg, respectively. Let $\mathscr{A}^H$ and $\mathscr{A}^{RH}$ be the atlas in the horocycles H and RH, that are the extended pushforwards of the atlases $\mathscr{A}$ and $R\mathscr{A}$, respectively. Let $i_H : \mathbb{S} \to H$ be the natural inclusion.

Let $h : \mathbb{S} \to R\mathbb{S}$ be the homeomorphism that conjugates g and Rg sending the marked point 0 of g in the marked point 0 of Rg.

Definition 2.2. The *Markov map* M_g *associated to* $g \in \mathscr{F}$ is the map $M_g : H \to H$ defined by

$$M_g(x) = \begin{cases} i_H \circ h^{-1} \circ i_{RS} \circ i_H^{-1}(x), \; if \; x \in i_H\,(S_0 \cup S_{a+1}) \\ i_H \circ h^{-1} \circ i_{RS} \circ g^{-k} \circ i_H^{-1}(x), \; if \; x \in i_H\,(S_k)\,, \; k = 1, \ldots, a \end{cases}.$$

We observe that, in particular, the *rigid Markov map* M_{r_γ} is an affine map with respect to the atlas $\mathscr{A}^H_{iso}$. Noting that $h(g^4(0)) = (Rg)^2(0) = g^2$ then M_g is as represented in Figure 2.4. We observe that the identification in H of $g(0)$ with $g^2(0)$ makes the Markov map M_g a local homeomorphism.

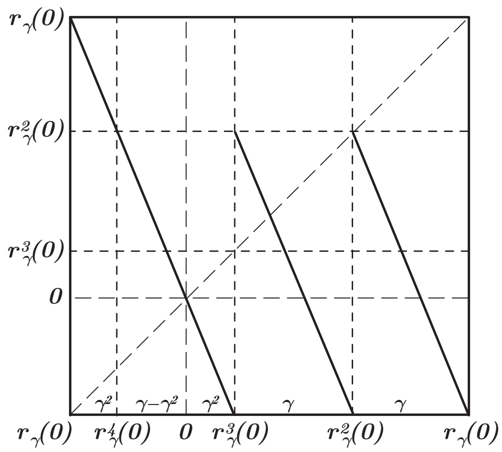

Fig. 2.4 The Markov map M_{r_γ} with respect to the atlas $\mathscr{A}^H_{iso}$.

Lemma 2.2. *Let $g \in \mathscr{F}$. The Markov map M_g associated to g is a C^{1+} local diffeomorphism with respect to the atlas $\mathscr{A}^H = (\pi_g)_* \mathscr{A}$ if, and only if, the diffeomorphism g is a fixed point of renormalization.*

The proof of Lemma 2.2 is in [21].

2.3 Anosov Diffeomorphisms

Fix a positive integer $a \in \mathbb{N}$ and consider the *Anosov automorphism* $G_a : \mathbb{T} \to \mathbb{T}$ given by $G_a(x,y) = (ax+y,x)$, where $\mathbb{T}$ is equal to $\mathbb{R}^2/(v\mathbb{Z} \times w\mathbb{Z})$ with $v = (\gamma,1)$ and $w = (-1,\gamma)$. Let $\pi : \mathbb{R}^2 \to \mathbb{T}$ be the natural projection. Let A_0 and B_0 be the rectangles $[0,1] \times [0,1]$ and $[-\gamma,0] \times [0,\gamma]$ respectively. A Markov partition $\mathscr{M}_{G_a}$ of G_a is given by $\mathbf{A} = \pi(A_0)$ and $\mathbf{B} = \pi(B_0)$ (see Figure 2.5). The unstable manifolds of G_a are the projection by π of the vertical lines in the plane, and the stable manifolds of G_a are the projection by π of the horizontal lines in the plane.

A C^{1+} *Anosov diffeomorphism* $G : \mathbb{T} \to \mathbb{T}$ is a $C^{1+\alpha}$ diffeomorphism, with $\alpha > 0$, such that (i) G is topologically conjugate to G_a; (ii) the tangent bundle has a $C^{1+\alpha}$ uniformly hyperbolic splitting into a stable direction and an unstable direction (see [39]). We denote by $\mathscr{G}$ the set of all such C^{1+} Anosov diffeomorphisms with an invariant measure absolutely continuous with respect to the Lebesgue measure.

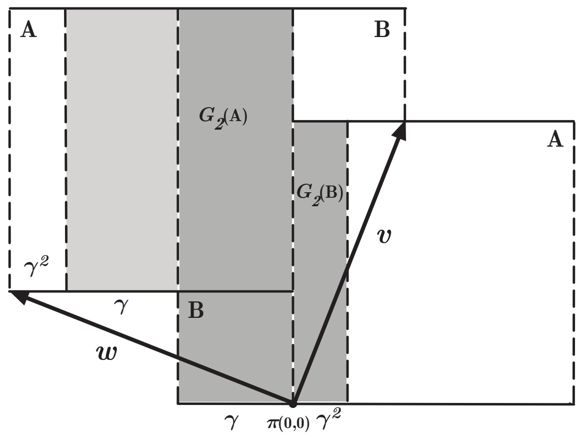

Fig. 2.5 The Anosov automorphism $G_2 : \mathbb{T} \to \mathbb{T}$.

If h is the topological conjugacy between G_a and G, then a Markov partition $\mathscr{M}_G$ of G is given by $h(\mathbf{A})$ and $h(\mathbf{B})$. Let $d = d_\rho$ be the distance on the torus $\mathbb{T}$, determined by a Riemannian metric ρ. We define the map $G_\iota = G$ if $\iota = u$, or $G_\iota = G^{-1}$ if $\iota = s$. For $\iota \in \{s,u\}$ and $x \in \mathbb{T}$, we denote the local ι-manifolds through x by

$$W^\iota(x,\varepsilon) = \left\{ y \in \mathbb{T} : d(G_\iota^{-n}(x), G_\iota^{-n}(y)) \leq \varepsilon, \text{ for all } n \geq 0 \right\}.$$

By the Stable Manifold Theorem (see [39]), these sets are respectively contained in the stable and unstable immersed manifolds

$$W^\iota(x) = \bigcup_{n\geq 0} G_\iota^n \left(W^\iota \left(G_\iota^{-n}(x), \varepsilon_0 \right) \right)$$

which are the image of $C^{1+\alpha}$ immersions $\kappa_{\iota,x} : \mathbb{R} \to \mathbb{T}$, for some $0 < \alpha \leq 1$ and some small $\varepsilon_0 > 0$. An *open* (resp. *closed*) *ι-leaf segment* I is defined as a subset of $W^\iota(x)$ of the form $\kappa_{\iota,x}(I_1)$ where I_1 is an open (resp. closed) subinterval (non-empty) in $\mathbb{R}$. An *ι-leaf segment* is either an open or closed ι-leaf segment. The *endpoints* of an ι-leaf segment $I = \kappa_{\iota,x}(I_1)$ are the points $\kappa_{\iota,x}(u)$ and $\kappa_{\iota,x}(v)$ where u and v are the endpoints of I_1. The *interior* of an ι-leaf segment I is the complement of its boundary. A map $c : I \to \mathbb{R}$ is an *ι-leaf chart* of an ι-leaf segment I if c is a homeomorphism onto its image.

2.3.1 Spanning Leaf Segments

One can find a small enough $\varepsilon_0 > 0$, such that for every $0 < \varepsilon < \varepsilon_0$ there is $\delta = \delta(\varepsilon) > 0$ with the property that, for all points $w, z \in \mathbb{T}$ with $d(w,z) < \delta$, $W^u(w,\varepsilon)$ and $W^s(z,\varepsilon)$ intersect in a unique point that we denote by

$$[w,z] = W^u(w,\varepsilon) \cap W^s(z,\varepsilon).$$

A *rectangle* R is a subset of $\mathbb{T}$ which is (i) closed under the bracket, i.e. $x, y \in R \Rightarrow [x,y] \in R$, and (ii) proper, i.e. it is the closure of its interior in $\mathbb{T}$. If ℓ^u and ℓ^s are respectively unstable and stable closed leaf segments intersecting in a single point then we denote by $[\ell^u, \ell^s]$ the set consisting of all points of the form $[w,z]$ with $w \in \ell^u$ and $z \in \ell^s$. We note that $[\ell^u, \ell^s]$ is a rectangle. Conversely, given a rectangle R, for each $x \in R$ there are closed unstable and stable leaf segments of $\mathbb{T}$, $\ell^u(x,R) \subset W^u(x)$ and $\ell^s(x,R) \subset W^s(x)$ such that $R = [\ell^u(x,R), \ell^s(x,R)]$. The leaf segments $\ell^u(x,R)$ and $\ell^s(x,R)$ are called, respectively, *unstable* and *stable spanning leaf segments*.

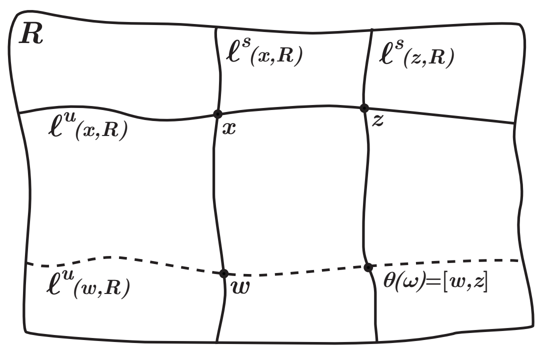

Fig. 2.6 A basic stable holonomy $\theta : \ell^u(x,R) \to \ell^u(z,R)$.

2.3.2 Basic Holonomies

Suppose that x and z are two points inside any rectangle R of $\mathbb{T}$. Let $\ell^s(x,R)$ and $\ell^s(z,R)$ be two stable spanning leaf segments of R containing, respectively, x and z. We define the map $\theta : \ell^s(x,R) \to \ell^s(z,R)$ by $\theta(w) = [w,z]$ (see Figure 2.6). Such maps are called the *basic stable holonomies*. They generate the pseudo-group of all stable holonomies. Similarly, we can define the *basic unstable holonomies*.

2.3.3 Lamination Atlas

The *stable lamination atlas* $\mathscr{L}^s = \mathscr{L}^s(G,\rho)$*, determined by a Riemannian metric* ρ, is the set of all maps $e : I \to \mathbb{R}$, where e is an isometry between the induced Riemannian metric on the stable leaf segment I and the Euclidean metric on the reals. We call the maps $e \in \mathscr{L}^s$ the *stable lamination charts*. Similarly, we can define the *unstable lamination atlas* $\mathscr{L}^u = \mathscr{L}^u(G,\rho)$. By Theorem 2.1 in [27], the basic unstable and stable holonomies are C^{1+} with respect to the lamination atlas $\mathscr{L}^s$.

2.3.4 Circle Diffeomorphisms

Let $G \in \mathscr{G}$ be a C^{1+} Anosov diffeomorphism with an invariant measure absolutely continuous with respect to the Lebesgue measure and topologically conjugate to the Anosov automorphism G_a by the homeomorphism h. For each Markov rectangle R, let t^s_R be the set of all unstable spanning leaf segments of R. Thus, by the local product structure, one can identify t^s_R with any stable spanning leaf segment $\ell^s(x,R)$ of R. We form the space $\mathbb{S}_G$ by taking the disjoint union $t^s_{h(\mathbf{A})} \bigsqcup t^s_{h(\mathbf{B})}$, where $h(\mathbf{A})$ and $h(\mathbf{B})$ are the Markov rectangles of the Markov partition $\mathscr{M}_G$, and identifying two points $I \in t^s_R$ and $J \in t^s_{R'}$ if (i) $R \neq R'$, (ii) the unstable leaf segments I and J are unstable boundaries of Markov rectangles, and (iii) $int(I \cap J) \neq \emptyset$. Topologically, the space $\mathbb{S}_G$ is a *counterclockwise oriented circle*. Let $\pi_{\mathbb{S}_G} : \bigsqcup_{R \in \mathscr{M}_G} R \to \mathbb{S}_G$ be the natural projection sending $x \in R$ to the point $\ell^u(x,R)$ in $\mathbb{S}_G$.

Let $I_\mathbb{S}$ be an arc of $\mathbb{S}_G$ and I a leaf segment such that $\pi_{\mathbb{S}_G}(I) = I_\mathbb{S}$. The chart $i : I \to \mathbb{R}$ in $\mathscr{L} = \mathscr{L}^s(G,\rho)$ determines a *circle chart* $i_\mathbb{S} : I_\mathbb{S} \to \mathbb{R}$ for $I_\mathbb{S}$ given by $i_\mathbb{S} \circ \pi_{\mathbb{S}_G} = i$. We denote by $\mathscr{A}_G = \mathscr{A}(G,\rho)$ the set of all circle charts $i_\mathbb{S}$ determined by charts i in $\mathscr{L} = \mathscr{L}^s(G,\rho)$. Given any circle charts $i_\mathbb{S} : I_\mathbb{S} \to \mathbb{R}$ and $j_\mathbb{S} : J_\mathbb{S} \to \mathbb{R}$, the overlap map $j_\mathbb{S} \circ i_\mathbb{S}^{-1} : i_\mathbb{S}(I_\mathbb{S} \cap J_\mathbb{S}) \to j_\mathbb{S}(I_\mathbb{S} \cap J_\mathbb{S})$ is equal to $j_\mathbb{S} \circ i_\mathbb{S}^{-1} = j \circ \theta \circ i^{-1}$, where $i = i_\mathbb{S} \circ \pi_{\mathbb{S}_G} : I \to \mathbb{R}$ and $j = j_\mathbb{S} \circ \pi_{\mathbb{S}_G} : J \to \mathbb{R}$ are charts in $\mathscr{L}$, and

$$\theta : i^{-1}(i_\mathbb{S}(I_\mathbb{S} \cap J_\mathbb{S})) \to j^{-1}(j_\mathbb{S}(I_\mathbb{S} \cap J_\mathbb{S}))$$

is a basic stable holonomy. By Theorem 2.1 in [27], there exists $\alpha > 0$ such that, for all circle charts $i_\mathbb{S}$ and $j_\mathbb{S}$ in $\mathscr{A}_G$, the overlap maps $j_\mathbb{S} \circ i_\mathbb{S}^{-1} = j \circ \theta \circ i^{-1}$ are $C^{1+\alpha}$ diffeomorphisms with a uniform bound in the $C^{1+\alpha}$ norm. Hence, $\mathscr{A}_G = \mathscr{A}(G,\rho)$ is a C^{1+} atlas.

Suppose that I and J are stable leaf segments and $\theta : I \to J$ is a holonomy map such that, for every $x \in I$, the unstable leaf segments with endpoints x and $\theta(x)$ cross once, and only once, a stable boundary of a Markov rectangle. We define the *arc rotation map* $\tilde{\theta}_G : \pi_{\mathbb{S}_G}(I) \to \pi_{\mathbb{S}_G}(J)$, associated to θ, by $\tilde{\theta}_G(\pi_{\mathbb{S}_G}(x)) = \pi_{\mathbb{S}_G}(\theta(x))$ (see Figure 2.7). By Theorem 2.1 in [27] there exists $\alpha > 0$ such that the holonomy $\theta : I \to J$ is a $C^{1+\alpha}$ diffeomorphism, with respect to the C^{1+} lamination atlas $\mathscr{L}^s(G,\rho)$. Hence, the arc rotation maps $\tilde{\theta}_G$ are C^{1+} diffeomorphisms, with respect to the C^{1+} atlas $\mathscr{A}(G,\rho)$.

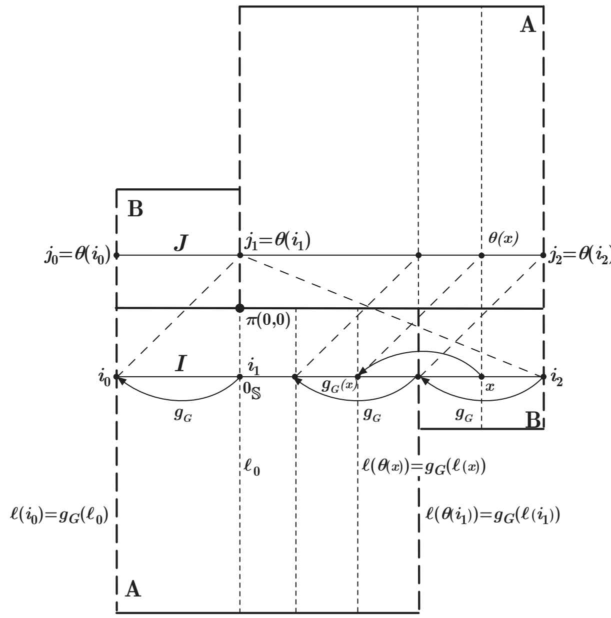

Fig. 2.7 The arc rotation map $g_G = \tilde{\theta}_G : \pi_{\mathbb{S}_G}(I) \to \pi_{\mathbb{S}_G}(J)$. We note that $\mathbb{S} = \pi_{\mathbb{S}_G}(I) = \pi_{\mathbb{S}_G}(J)$ and $\ell(x) = \pi_{\mathbb{S}_G}(x)$ is the unstable spanning leaf segment containing x.

Lemma 2.3. *There is a well-defined C^{1+} circle diffeomorphism g_G, with respect to the C^{1+} atlas $\mathscr{A}_G = \mathscr{A}(G,\rho)$, such that $g_G|\pi_{\mathbb{S}_G}(I) = \tilde{\theta}_G$, for every arc rotation map $\tilde{\theta}_G$. In particular, if G_a is the Anosov automorphism, then g_{G_a} is the rigid rotation r_γ, with respect to the isometric atlas $\mathscr{A}_{iso} = \mathscr{A}(G_a,E)$, where E corresponds to the Euclidean metric in the plane.*

The proof of Lemma 2.3 is in [21].

2.3.5 Train-Tracks and Markov Maps

Roughly speaking, train-tracks are the optimal leaf-quotient spaces on which the stable and unstable Markov maps induced by the action of G on leaf segments are local homeomorphisms.

Let $G \in \mathscr{G}$ be a C^{1+} Anosov diffeomorphism. Let h be the homeomorphism that conjugates G with G_a. We recall that, for each Markov rectangle R, t_R^s denotes the set of all unstable spanning leaf segments of R and, by the local product structure, one can identify t_R^s with any stable spanning leaf segment $\ell^s(x,R)$ of R. We form the space $\mathbf{T}_G$ by taking the disjoint union $t^s_{h(\mathbf{A})} \bigsqcup t^s_{h(\mathbf{B})}$, where $h(\mathbf{A})$ and $h(\mathbf{B})$ are the

Markov rectangles of the Markov partition $\mathscr{M}_G$ and identifying two points $I \in t^s_R$ and $J \in t^s_{R'}$ if (i) the unstable leaf segments I and J are unstable boundaries of Markov rectangles and (ii) $int(I \cap J) = \emptyset$. This space is called the *stable train-track* and it is denoted by $\mathbf{T}_G$.

Let $\pi_{\mathbf{T}_G} : \bigsqcup_{R \in \mathscr{M}_G} R \to \mathbf{T}_G$ be the natural projection sending the point $x \in R$ to the point $\ell^u(x,R)$ in $\mathbf{T}_G$. A *topologically regular point* I in $\mathbf{T}_G$ is a point with a unique preimage under $\pi_{\mathbf{T}_G}$ (i.e. the preimage of I is not a union of distinct unstable boundaries of Markov rectangles). If a point has more than one preimage by $\pi_{\mathbf{T}_G}$, then we call it a *junction*. Hence, there is only one junction.

A chart $i : I \to \mathbb{R}$ in $\mathscr{L} = \mathscr{L}^s(G,\rho)$ determines a *train-track chart* $i_{\mathrm{T}} : I_T \to \mathbb{R}$ for I_T given by $i_T \circ \pi_{\mathbf{T}_G} = i$. We denote by $\mathscr{A}^{\mathbf{T}_G} = \mathscr{A}^{\mathbf{T}_G}(G,\rho)$ the set of all train-track charts i_T determined by charts i in $\mathscr{L} = \mathscr{L}^s(G,\rho)$. Given any train-track charts $i_T : I_T \to \mathbb{R}$ and $j_T : J_T \to \mathbb{R}$ in $\mathscr{A}^{\mathbf{T}_G}$, the overlap map $j_T \circ i_T^{-1} : i_T(I_T \cap J_T) \to j_T(I_T \cap J_T)$ is equal to $j_T \circ i_T^{-1} = j \circ \theta \circ i^{-1}$, where $i = i_T \circ \pi_{\mathbf{T}_G} : I \to \mathbb{R}$ and $j = j_T \circ \pi_{\mathbf{T}_G} : J \to \mathbb{R}$ are charts in $\mathscr{L}$, and

$$\theta : i^{-1}(i_T(I_T \cap J_T)) \to j^{-1}(j_T(I_T \cap J_T))$$

is a basic stable holonomy. By Theorem 2.1 in [27] there exists $\alpha > 0$ such that, for all train-track charts i_T and j_T in $\mathscr{A}^{\mathbf{T}_G}(G,\rho)$, the overlap maps $j_T \circ i_T^{-1} = j \circ \theta \circ i^{-1}$ have $C^{1+\alpha}$ diffeomorphic extensions with a uniform bound in the $C^{1+\alpha}$ norm. Hence, $\mathscr{A}^{\mathbf{T}_G}(G,\rho)$ is a $C^{1+\alpha}$ atlas in $\mathbf{T}_G$.

The *(stable) Markov map* $M_G : \mathbf{T}_G \to \mathbf{T}_G$ is the mapping induced by the action of G on unstable spanning leaf segments, that it is defined as follows: if $I \in \mathbf{T}_G$, $M_G(I) = \pi_{\mathbf{T}_G}(G(I))$ is the unstable spanning leaf segment containing $G(I)$. This map M_G is a local homeomorphism because G sends short stable leaf segments homeomorphically onto short stable leaf segments.

A *stable leaf primary cylinder of a Markov rectangle* R is a stable spanning leaf segment of R. For $n \geq 1$, a *stable leaf n-cylinder of* R is a stable leaf segment I such that (i) $G^n I$ is a stable leaf primary cylinder of a Markov rectangle $R'(I) \in \mathscr{M}_G$; (ii) $G^n(\ell^u(x,R)) \subset R'(I)$ for every $x \in I$, where $\ell^u(x,R)$ is an unstable spanning leaf segment of R.

For $n \geq 1$, an *n-cylinder* is the projection into $\mathbf{T}_G$ of a stable leaf n-cylinder segment. Thus, each Markov rectangle in $\mathbb{T}$ projects in a unique primary stable leaf segment in $\mathbf{T}_G$.

Given a topological chart (e,U) on the train-track $\mathbf{T}_G$ and a train-track segment $C \subset U$, we denote by $|C|_e$ the length of $e(C)$. We say that M_G has *bounded geometry* in a C^{1+} atlas $\mathscr{A}$, if there is $\kappa_1 > 0$ such that, for every n-cylinder C_1 and n-cylinder C_2 with a common endpoint with C_1, we have $\kappa_1^{-1} < |C_1|_e/|C_2|_e < \kappa_1$, where the lengths are measured in any chart (e,U) of the atlas such that $C_1 \cup C_2 \subset U$. We note that M_G has bounded geometry, with respect to a C^{1+} atlas $\mathscr{A}$, if, and only if, there are $\kappa_2 > 0$ and $0 < \nu < 1$ such that $|C|_e \leq \kappa_2 \nu^n$, for every n-cylinder and every $e \in \mathscr{B}$.

By Section 4.3 in Pinto-Rand [25], we obtain that M_G is C^{1+} and has *bounded geometry* in $\mathscr{A}^{\mathbf{T}_G}(G,\rho)$. In [21] it is proved that M_G is the Markov map M_{g_G}, as in

Definition 2.2. Hence, g_G is a fixed point of renormalization. Futhermore, in [21] it is proved that the map $G \to g_G$ induces an one-to-one correspondence between C^{1+} conjugacy classes of Anosov diffeomorphisms with an invariant measure absolutely continuous with respect to the Lebesgue measure and C^{1+} conjugacy classes of circle diffeomorphisms that are fixed points of renormalization.

We recall that $\mathscr{G}$ denotes the set of C^{1+} Anosov diffeomorphisms G that are topologically conjugate to the linear Anosov automorphism G_a and have an invariant measure absolutely continuous with respect to the Lebesgue measure.

Theorem 2.1. *For every $G \in \mathscr{G}$, the C^{1+} circle diffeomorphism g_G is a C^{1+} fixed point of renormalization, with respect to the C^{1+} atlas $\mathscr{A}_G = \mathscr{A}(G,\rho)$.*

The proof of Theorem 2.1 is in [21].

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